CMPUT 399

Intro Robotics & Mechatronics:

Statics & Dynamics

Some slides are taken from Prof. O. Khatib - lecture notes (Introduction to Robotics)
Some slides are taken from Prof. A. De Luca - lecture notes (Robotics 1)

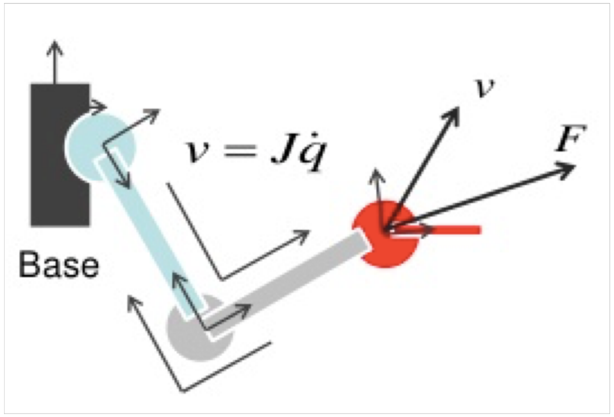
CASSIE biped robot

- CASSIE, a biped robot created by Agility Robotics (spin-off Oregon St. University)
  - Balance while walking
  - Interact with unknown environment
  - uses **sensors** embedded in its **legs** to keep its balance.
- Robots move from industry to **human environments** which is
  - **Unknown** (time varying) with lots of uncertainties
- Interaction with human -> **safety** -> **control**
  - **Statics** (force balance) >> **Dynamics** (force & motion)

Kinematics (review)

- Kinematics: $\dot{x} = J\,\dot{q}$

  (relation between q , x)

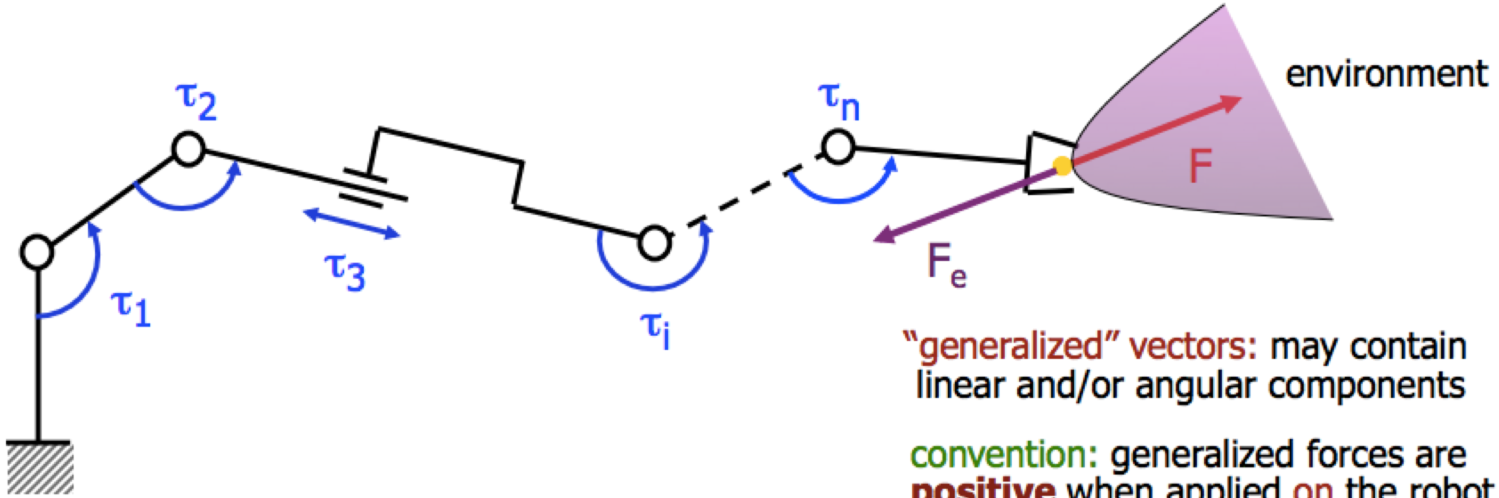


- Inverse Kinematics:

when J **square** and **non-singular** → $\dot{q} = J^{-1}(q)\,\dot{x}$

  - near **singularity** of the Jacobian matrix (high $\dot{q}$)!
  - for redundant robots (n≠m), no standard “inverse” of a rectangular matrix → pseudo-inverse.

Generalized Forces and Torques



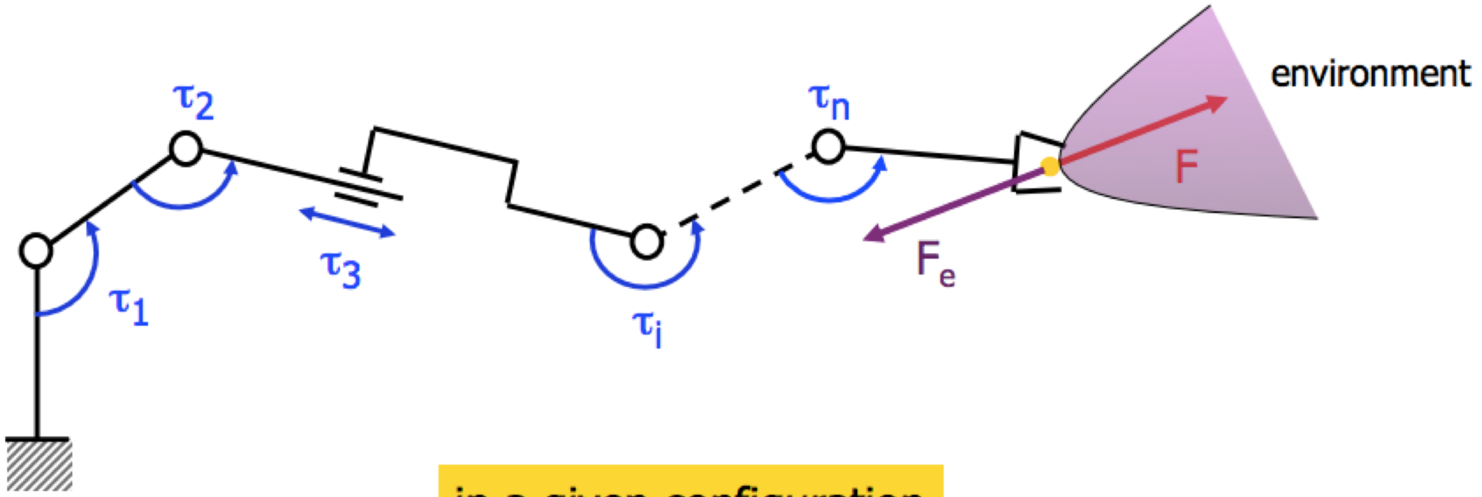
"generalized" vectors: may contain linear and/or angular components

convention: generalized forces are **positive** when applied on the robot

- τ = forces/torques exerted by the motors at the robot joints
- F = equivalent forces/torques exerted at the robot end-effector
- F_e = forces/torques exerted by the environment at the end-effector
- principle of action and reaction: $F_e = - F$

reaction from environment is equal and opposite to the robot action on it

Statics - Transformation of Forces



in a given configuration

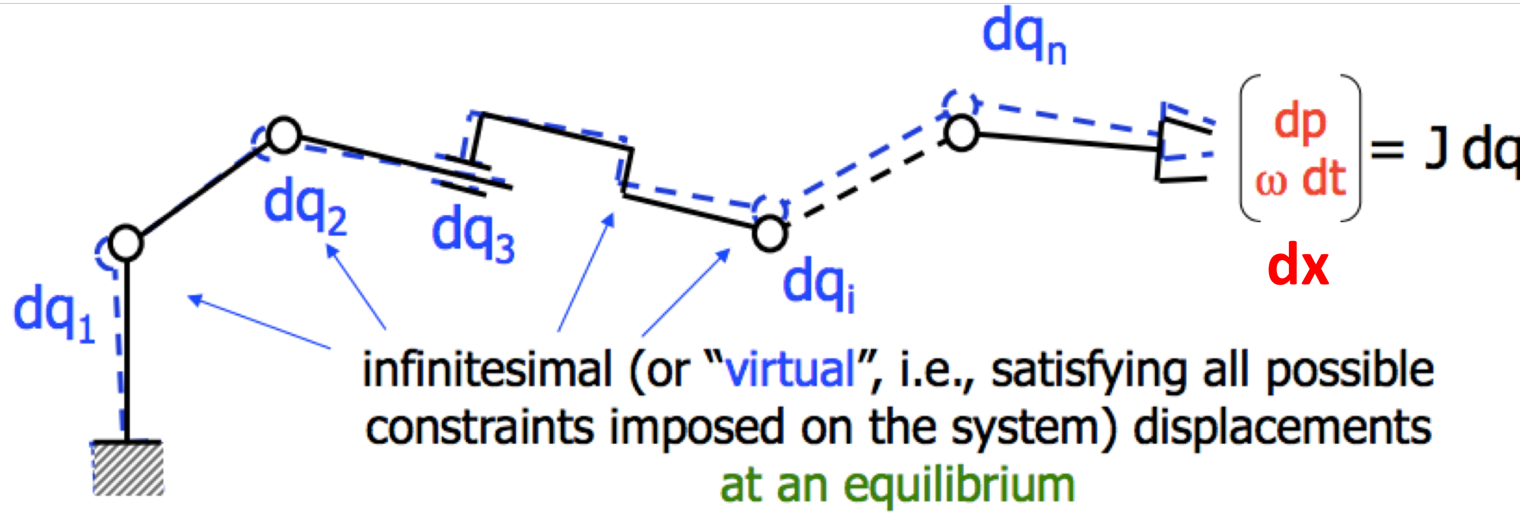
- what is the transformation between F at robot end-effector and τ at joints?

in **static equilibrium** conditions (i.e., **no motion**):

- what F will be exerted on environment by a τ applied at the robot joints?
- what τ at the joints will balance a F_e (= -F) exerted by the environment?

all equivalent formulations

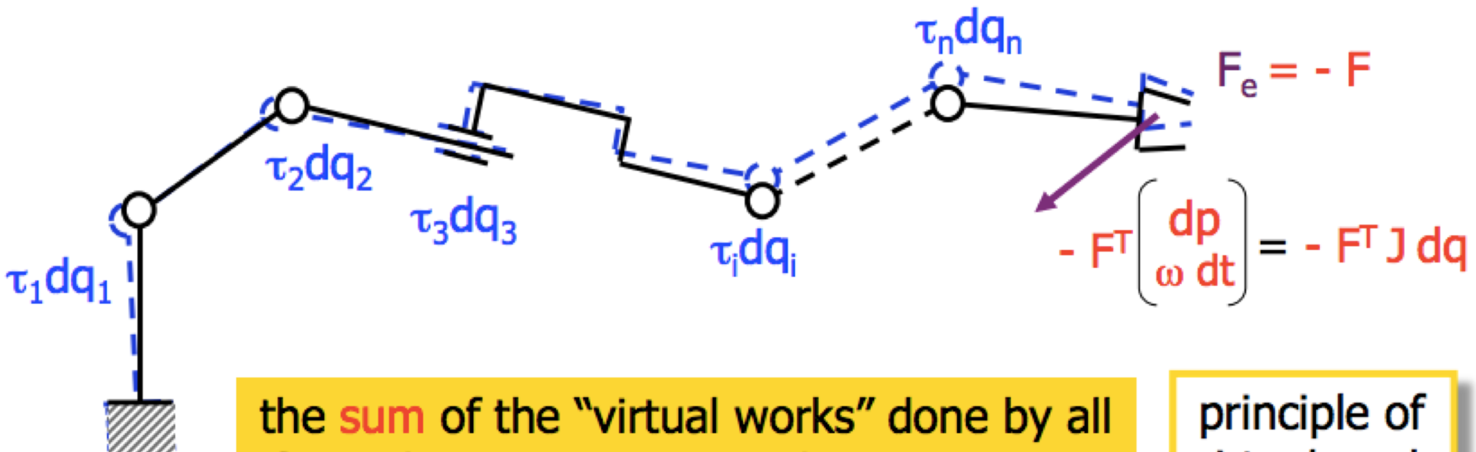
Virtual displacement and works



- without kinetic energy variation (zero acceleration)
- without dissipative effects (zero velocity)

the "virtual work" is the work done by all forces/torques acting on the system for a given virtual displacement

Principle of Virtual Work



the sum of the "virtual works" done by all forces/torques acting on the system = 0

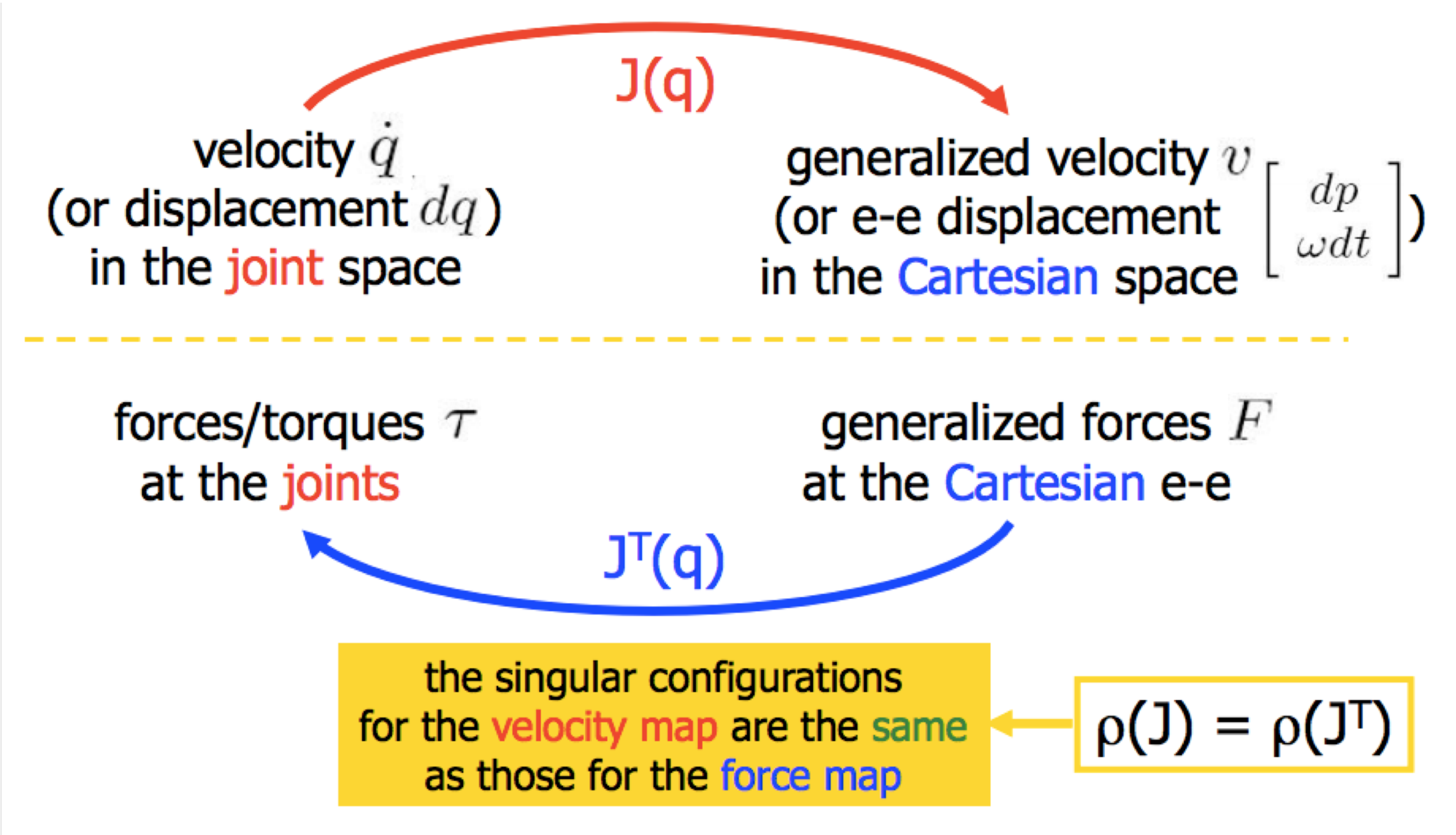
principle of virtual work

$$\tau^T dq - F^T \begin{bmatrix} dp \\ \omega dt \end{bmatrix} = \tau^T dq - F^T J dq = 0 \quad \forall dq$$

$$\Rightarrow \quad \tau = J^T(q) F$$

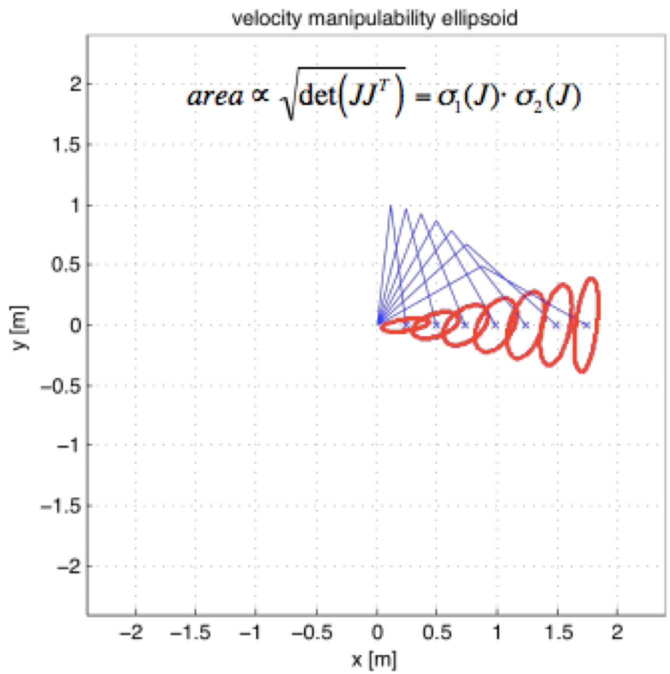
At ***static equilibrium***, the virtual work done by active forces is **zero**.

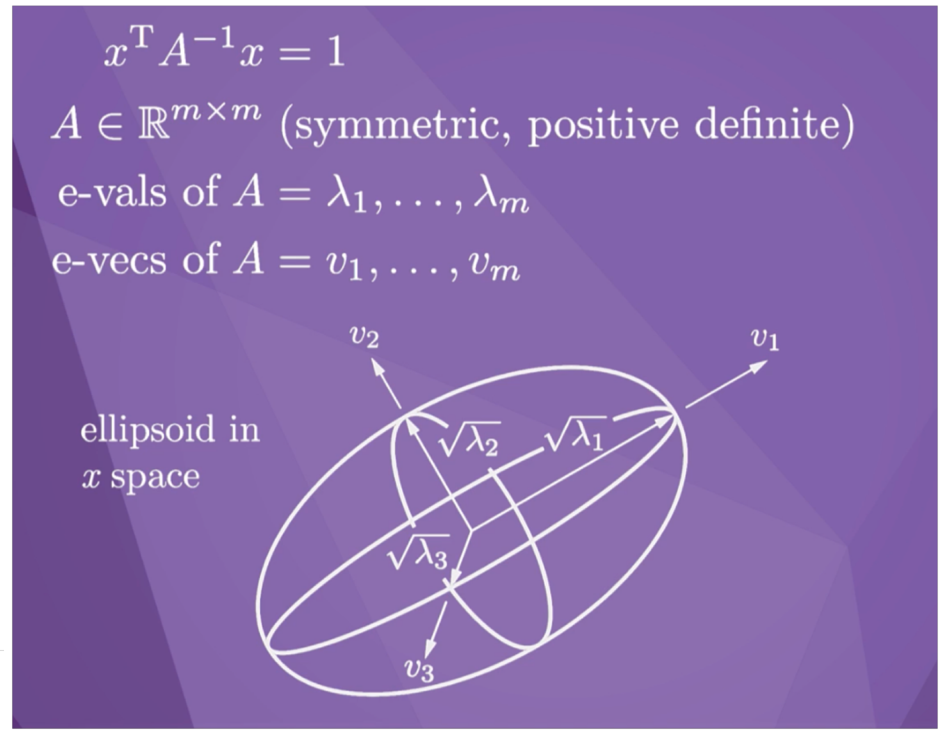
Duality between Velocity & Force



Duality between Velocity & Force

planar 2R arm with unitary links





$$\dot{q}^T \dot{q} = 1 \quad \Rightarrow \quad v^T J^{\#T} J^{\#} v = 1$$

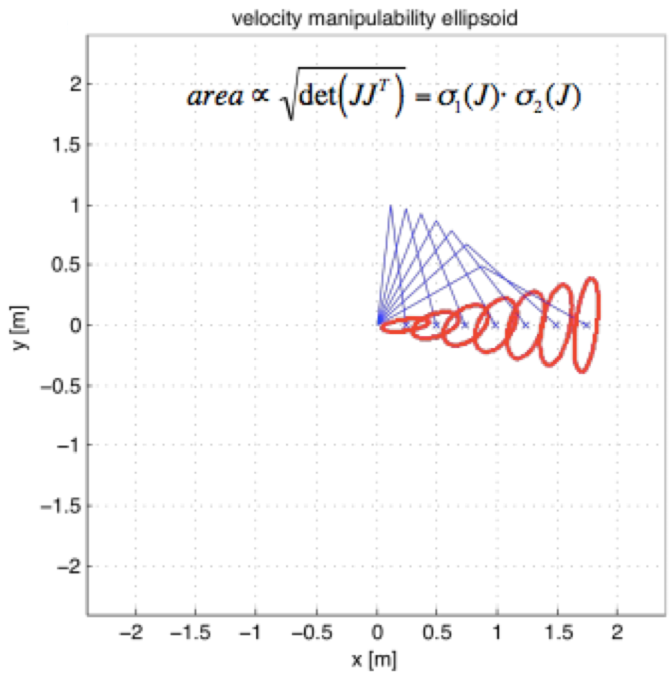
task velocity manipulability ellipsoid

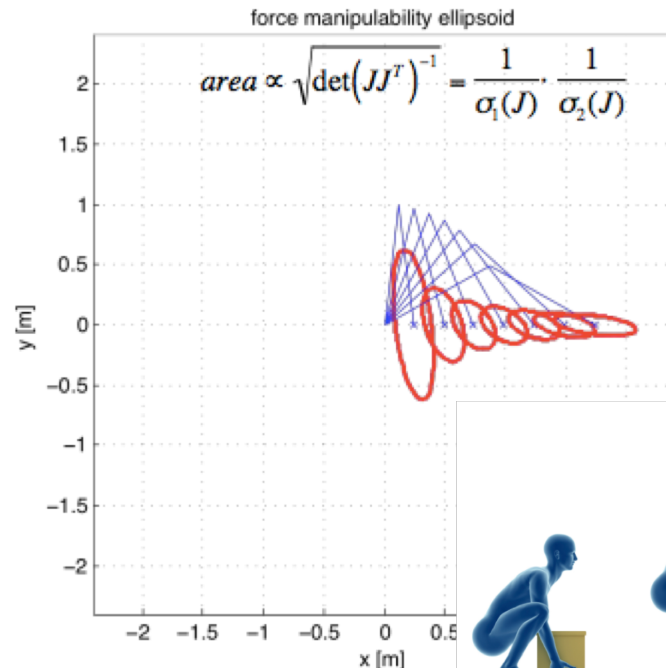
$(JJ^T)^{-1}$ if $\rho = m$

"how easily" can the end-effector be **moved** in the various directions of the task space

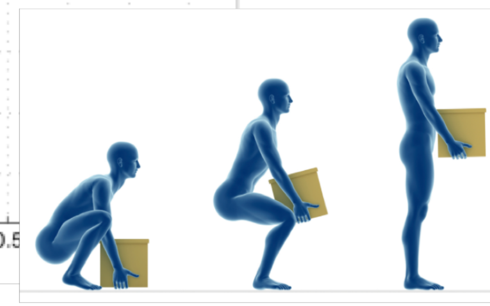
Duality between Velocity & Force

planar 2R arm with unitary links





@ **singularity** robot cannot move in certain direction but can sustain a lot of forces (by the structure of mechanism)



$\dot{q}^T \dot{q} = 1 \Rightarrow v^T J^{\#T} J^{\#} v = 1$

task velocity manipulability ellipsoid

$J^{\#T} J^{\#} = (JJ^T)^{-1}$ if ρ = m

$\tau^T \tau = 1 \Rightarrow F^T J J^T F = 1$

same directions of the principal axes of the velocity ellipsoid, but with semi-axes of inverse lengths → task force manipulability ellipsoid

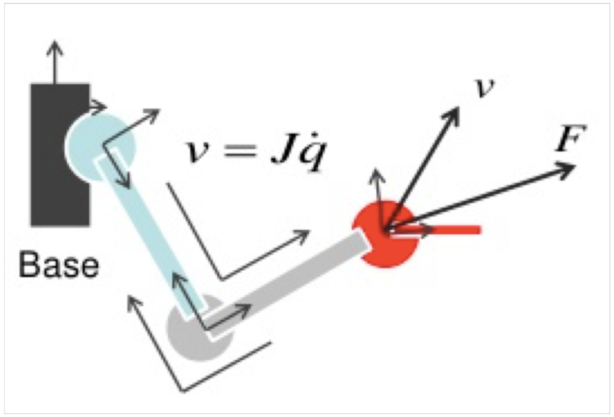
"how easily" can the end-effector be **moved** in the various directions of the task space

"how easily" can the end-effector apply **forces** (or balance applied ones) in the various directions of the task space

Kinematics - Statics Duality

- Kinematics: $\dot{x} = J\,\dot{q}$

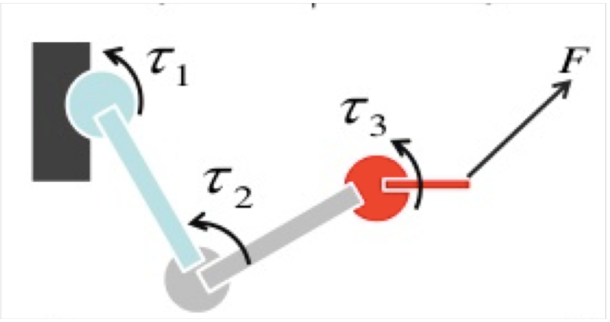
  (relation between q , x)



- Statics:

  (relation between τ and F)

  $$\tau = J^T\,F$$

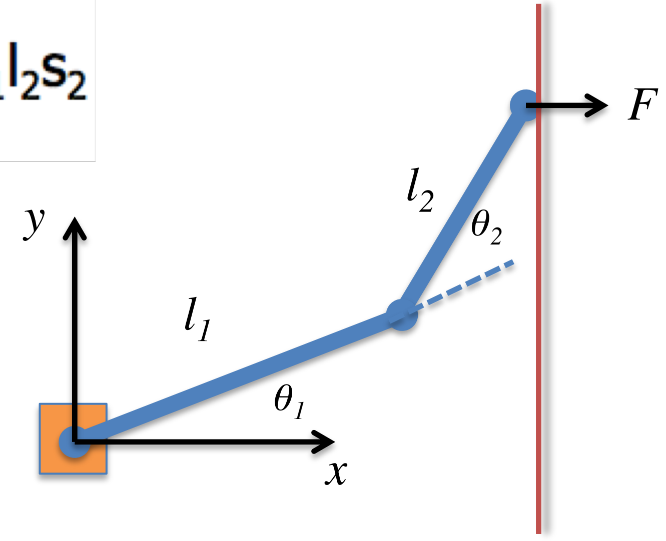


Example 1

- What are the joint torques required for the 2-link robot to push against the wall in x-direction?

$$J(q) = \begin{pmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{pmatrix} \quad \det J(q) = l_1 l_2 s_2$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = J^T \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

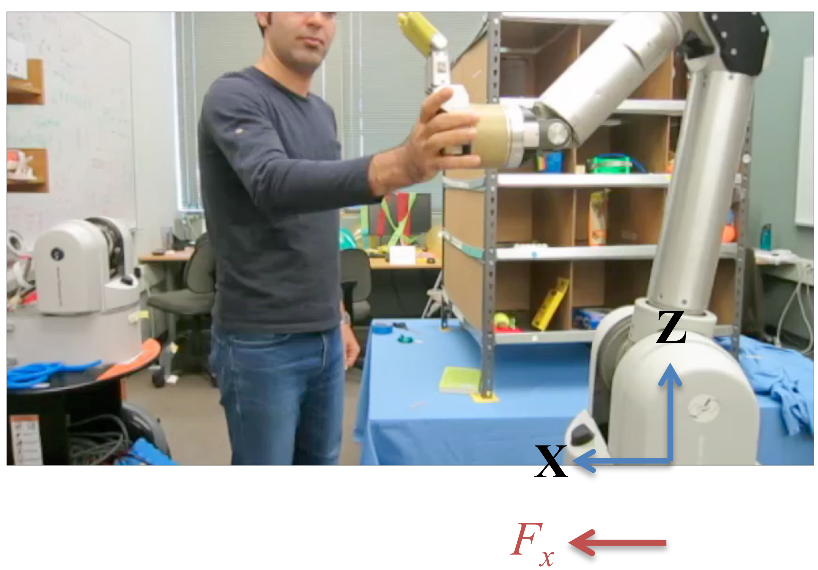


Example 2: WAM robot video

- WAM robot: applying force/moment in all 6-dofs

$$\tau = \tau_g + J_{lin}^T F_{base} + J_{rot}^T M_{base}$$

$$\dot{x} = \begin{bmatrix} J_{lin} \\ -- \\ J_{rot} \end{bmatrix} \dot{q}$$

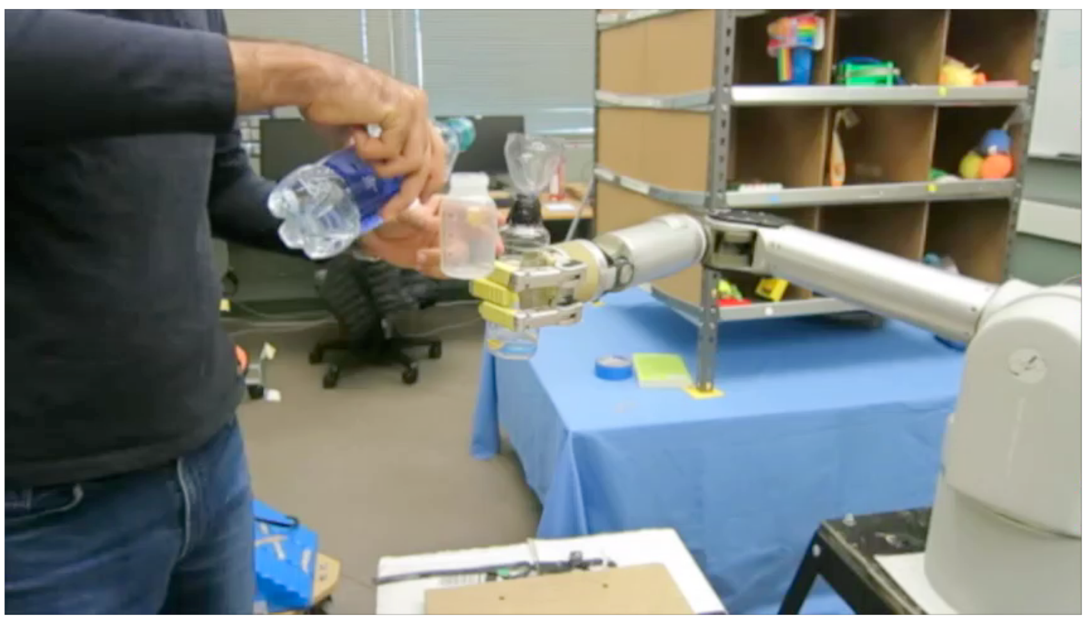


Example 3: estimation of external forces form joint torques measurement

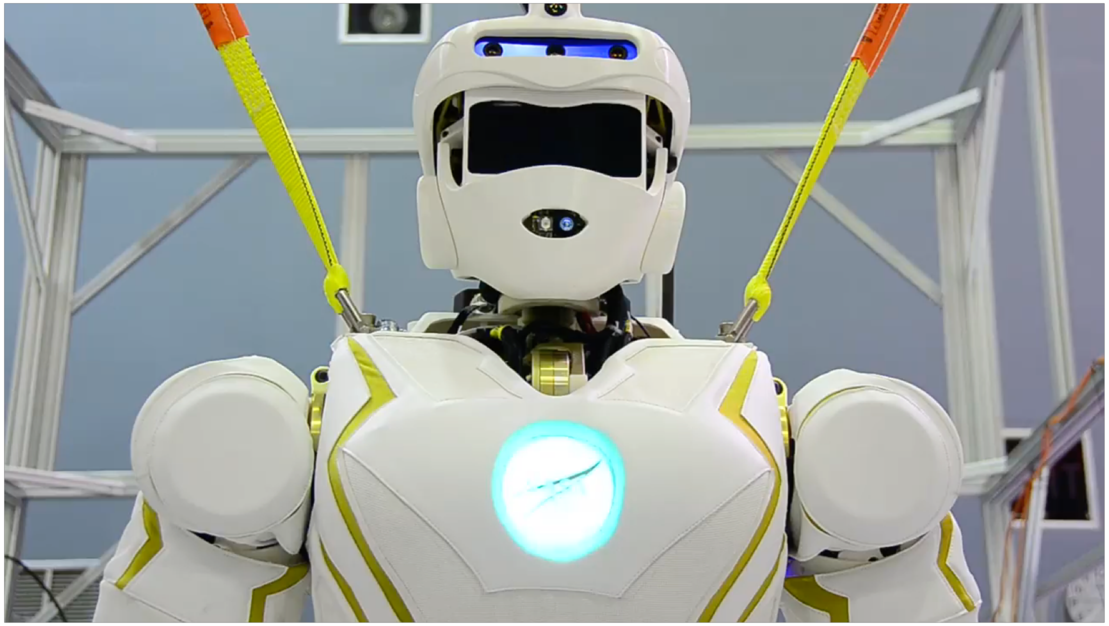
- In static condition:

$$J_c^T F_c = \underbrace{\tau_{wam}}_{\texttt{wam.getJointTorque()}} - \underbrace{\tau_g}_{\texttt{bt\_calgrav\_eval}}$$

$$F_c = J_c^{\dagger}(\tau_{wam} - \tau_g) \qquad J_c^{\dagger} = (J_c J_c^T)^{-1} J_c$$



Valkyrie NASA robot



Dynamics

Dynamics: relates **forces/torques** and **motion (in joint space or workspace variables)**

- Given motion variables (e.g. $\theta, \dot{\theta}, \ddot{\theta}$ or $x\ \dot{x}, \ddot{x}$), what joint torques (τ) or end-effector forces (f) would have been the cause?
  (this is inverse dynamics)
- Given joint torques (τ) or end-effector forces (f), what motions (e.g. $\theta, \dot{\theta}, \ddot{\theta}$ or $x\ \dot{x}, \ddot{x}$) would result? (this is forward dynamics)

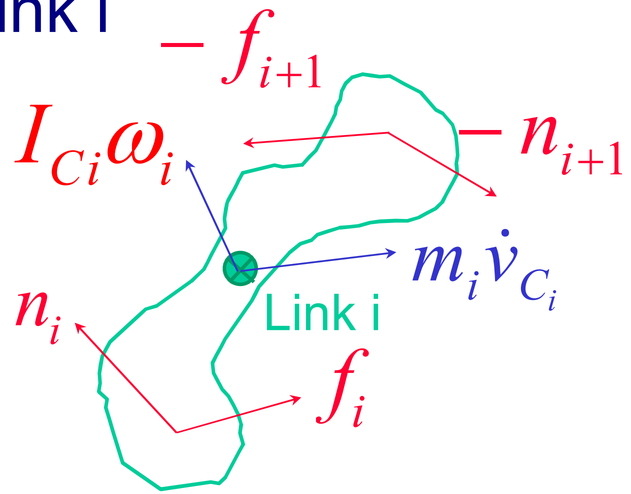
$$M(q)\ddot{q} + N(q, \dot{q})\dot{q} + g(q) = \tau$$

- Need to understand rigid body dynamics
  - **Newton – Euler method**
  - **Lagrange formulation (energy-based)**

Rigid body dynamics (linear + angular)

Dynamic forces on Link i

$$I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i$$



$$m_i \dot{\mathbf{v}}_{C_i} = \sum forces$$

$$I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i = \sum moments / c_i$$

Inertial forces/moments

$$F_i = m_i \dot{\mathbf{v}}_{C_i}$$ Linear motion: (Newton Eqn)

$$N_i = I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i$$ Angular motion: (Euler Eqn)

Lagrange formulation (energy based)

Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \tau$$

Lgrangian → $L = K - U$ (K: Kinetic Energy; U: Potential Energy)

Since $U = U(q)$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} + \frac{\partial U}{\partial q} = \tau$$

Inertial forces: $\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q}$

Gravity vector: $\frac{\partial U}{\partial q}$

Inertial forces

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} = \tau - G \qquad K = \frac{1}{2}\dot{q}^T M(q)\dot{q}$$

$$\frac{\partial K}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}\left[\frac{1}{2}\dot{q}^T M(q)\dot{q}\right] = M(q)\dot{q}$$

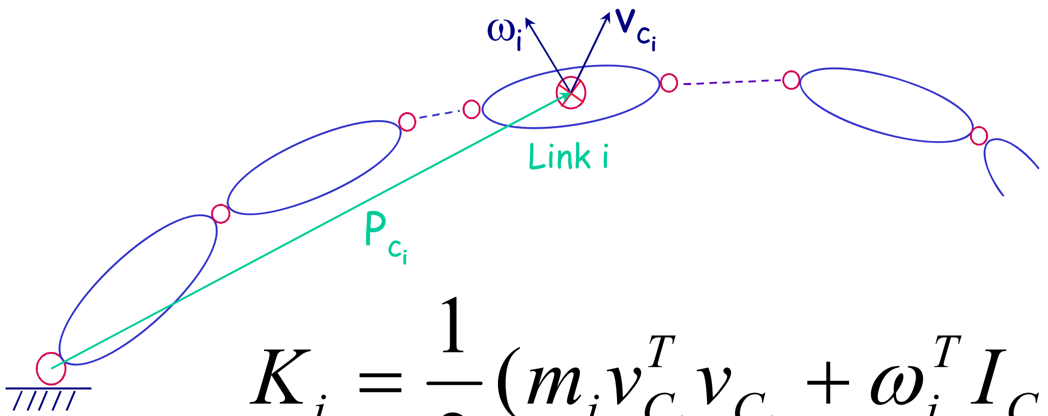
$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) = \frac{d}{dt}(M\dot{q}) = M\ddot{q} + \dot{M}\dot{q}$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} = M\ddot{q} + \dot{M}\dot{q} - \frac{1}{2}\begin{bmatrix} \dot{q}^T \frac{\partial M}{\partial q_1}\dot{q} \\ \vdots \\ \dot{q}^T \frac{\partial M}{\partial q_n}\dot{q} \end{bmatrix} = M\ddot{q} + V(q,\dot{q})$$

Kinetic energy of a link

Equations of Motion

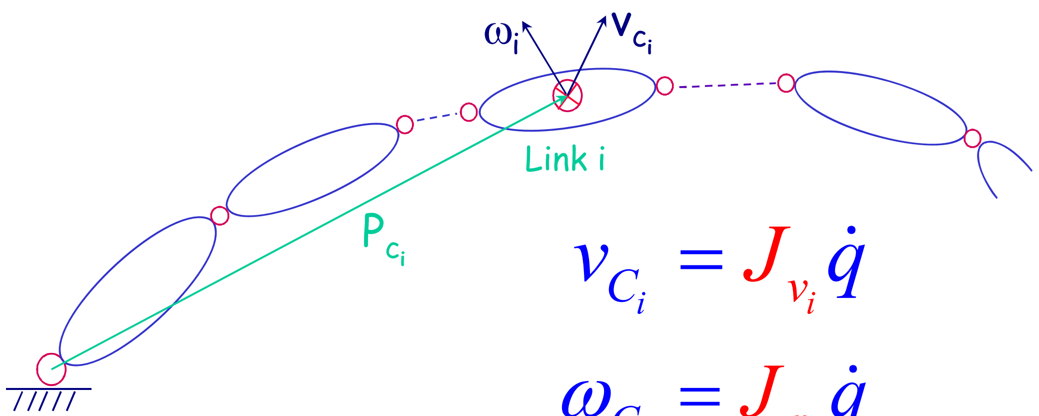
Explicit Form



$$K_i = \frac{1}{2}(m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{C_i} \omega_i)$$

Total Kinetic Energy ⇨ $K = \sum_{i=1}^{n} K_i$

Equations of Motion — Explicit Form



$$v_{C_i} = J_{v_i}\dot{q}$$

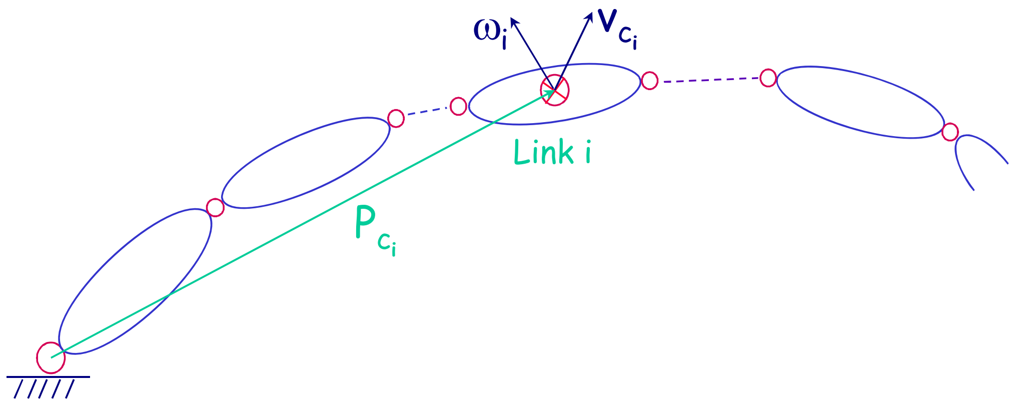
$$\omega_{C_i} = J_{\omega_i}\dot{q}$$

$$\frac{1}{2}\dot{q}^T M \dot{q} = \frac{1}{2}\sum_{i=1}^{n}(m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{C_i}\omega_i)$$

$$= \frac{1}{2}\sum_{i=1}^{n}(m_i \dot{q}^T J_{v_i}^T J_{v_i}\dot{q} + \dot{q}^T J_{\omega_i}^T I_{Ci} J_{\omega_i}\dot{q})$$

Equations of Motion

Explicit Form



$$\frac{1}{2}\dot{q}^T M \dot{q} = \frac{1}{2}\dot{q}^T \left[\sum_{i=1}^{n} (m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{Ci} J_{\omega_i}) \right] \dot{q}$$

$$M = \sum_{i=1}^{n} (m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{Ci} J_{\omega_i})$$

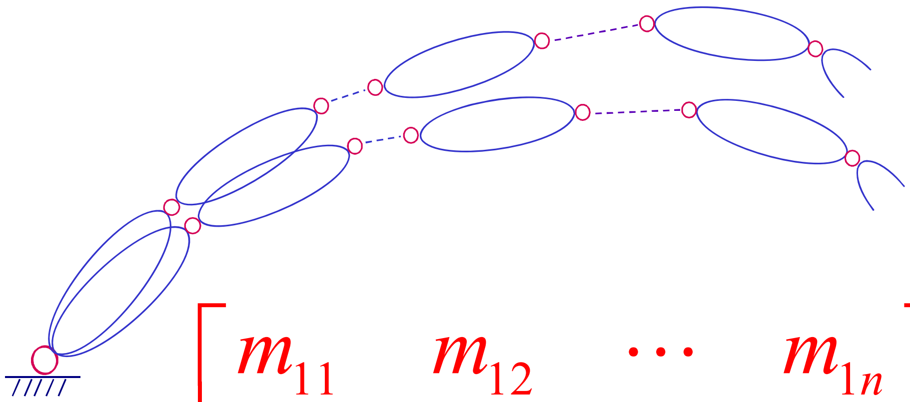
Dynamic Simulation

Cup Grasping and Handling

https://www.youtube.com/watch?v=MnY8z1D0xiU&feature=youtu.be

$$M(q)\ddot{q} + N(q, \dot{q})\dot{q} + g(q) = \tau$$

Properties of $\boldsymbol{M(q)}$ matrix

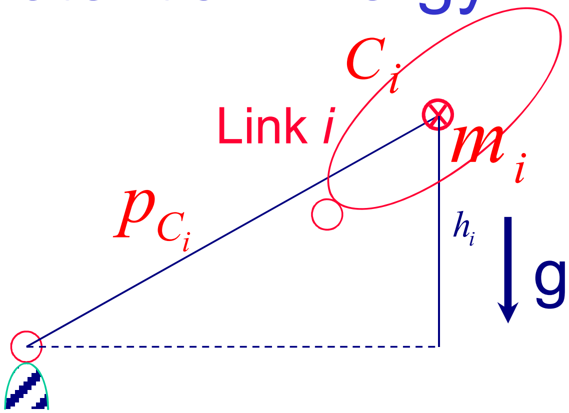


$$\underset{(n\times n)}{M(q)} = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}$$

$\boldsymbol{M(q)}$ *is Symmetric:* $\quad \boldsymbol{M} = \boldsymbol{M}^T \quad i.e.\ m_{12} = m_{21}, \ldots$

$\boldsymbol{M(q)}$ *is Positive definite:* $\quad \xi^T \boldsymbol{M(q)}\, \xi > 0 \quad \text{for all } \xi \neq 0$

Potential Energy



$$U_i = m_i g_0 h_i + U_0$$

$$U_i = m_i(-g^T p_{C_i}) + U_0$$

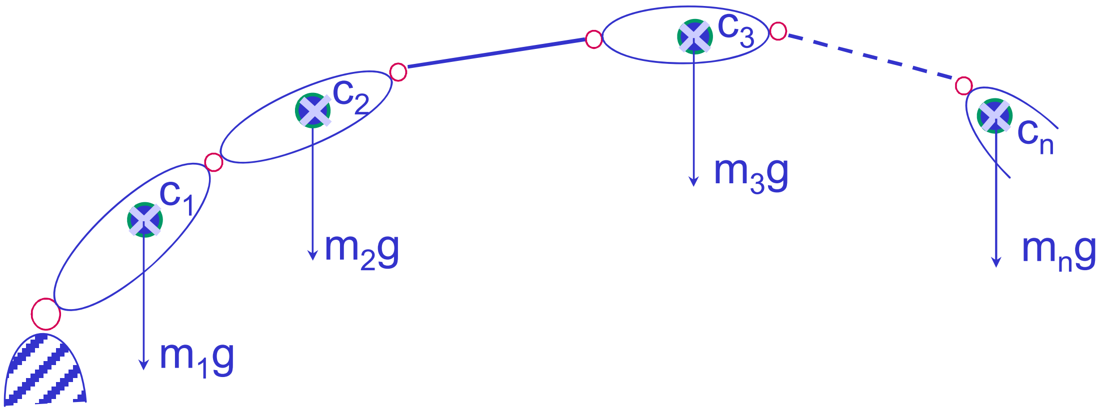
$$U = \sum U_i$$

Gravity Vector

$$G_j = \frac{\partial U}{\partial q_j} = -\sum_{i=1}^{n} (m_i g^T \frac{\partial \mathbf{p}_{C_i}}{\partial q_j})$$

$$G = -\begin{pmatrix} J_{v_1}^T & J_{v_2}^T & \cdots & J_{v_n}^T \end{pmatrix} \begin{pmatrix} m_1 g \\ m_2 g \\ \vdots \\ m_n g \end{pmatrix}$$

Gravity Vector



$$G = -\left(J_{v_1}^T (m_1 g) + J_{v_2}^T (m_2 g) + \cdots + J_{v_n}^T (m_n g)\right)$$

Example: 1-Rev. Joint + 1-prismatic

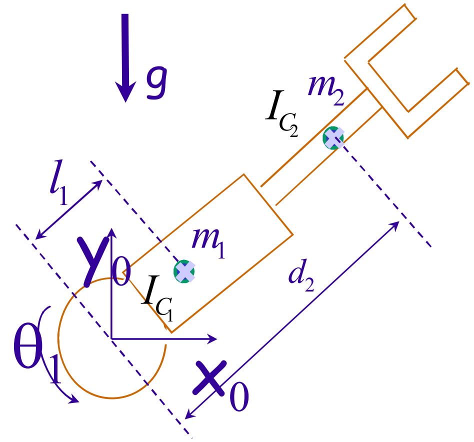
taken from Prof. O. Khatib lecture notes
(CS223A - Introduction to Robotics)
https://see.stanford.edu/Course/CS223A

Matrix M

$$M = m_1 J_{v_1}^T J_{v_1} + J_{\omega_1}^T I_{C_1} J_{\omega_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_2}^T I_{C_2} J_{\omega_2}$$

J_{v_1} and J_{v_2} : direct differentiation of the vectors:

$$^0\mathbf{p}_{C_1} = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix}; \text{ and } ^0\mathbf{p}_{C_2} = \begin{bmatrix} d_2 c_1 \\ d_2 s_1 \\ 0 \end{bmatrix}$$



In frame {0}, these matrices are:

$$^0J_{v_1} = \begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } ^0J_{v_2} = \begin{bmatrix} -d_2 s_1 & c_1 \\ d_2 c_1 & s_1 \\ 0 & 0 \end{bmatrix}$$

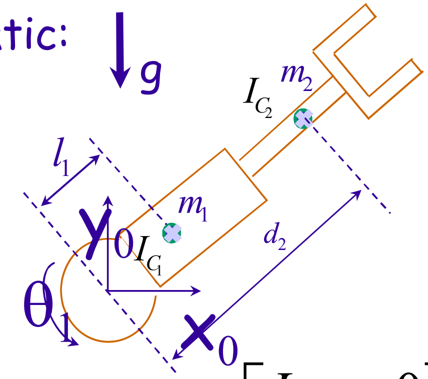
This yields

$$m_1 ({}^0J_{v_1}^T \, {}^0J_{v_1}) = \begin{bmatrix} m_1 l_1^2 & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } m_2 ({}^0J_{v_2}^T \, {}^0J_{v_2}) = \begin{bmatrix} m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix}$$

The matrices J_{ω_1} and J_{ω_2} are given by

$$J_{\omega_1} = \left[\overline{\epsilon}_1 \mathbf{z}_1 \quad 0\right] = \text{ and } J_{\omega_2} = \left[\overline{\epsilon}_1 \mathbf{z}_1 \quad \overline{\epsilon}_2 \mathbf{z}_2\right]$$

Joint 1 is revolute and joint 2 is prismatic:



$${}^1J_{\omega_1} = {}^1J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

And

$$({}^1J_{\omega_1}^T \, {}^1I_{C_1} \, {}^1J_{\omega_1}) = \begin{bmatrix} I_{zz1} & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } ({}^1J_{\omega_2}^T \, {}^1I_{C_2} \, {}^1J_{\omega_2}) = \begin{bmatrix} I_{zz2} & 0 \\ 0 & 0 \end{bmatrix}$$

Finally,

$$M = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix}$$

Vector $V(\mathbf{q},\dot{\mathbf{q}})$

$$\frac{\partial M}{\partial q_1} \qquad \frac{\partial m_{11}}{\partial q_1}$$

$$V = \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T M_{q_1}\dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T M_{q_2}\dot{\mathbf{q}} \end{bmatrix} = \begin{pmatrix} \dot{m}_{11} & \dot{m}_{12} \\ \dot{m}_{12} & \dot{m}_{22} \end{pmatrix}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \begin{pmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{pmatrix}\dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T \begin{pmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{pmatrix}\dot{\mathbf{q}} \end{bmatrix}$$

$$\dot{m}_{ij} = m_{ij1}\dot{q}_1 + m_{ij2}\dot{q}_2 \qquad \frac{\partial m_{22}}{\partial q_2}$$

$$V(\mathbf{q},\dot{\mathbf{q}}) = \begin{bmatrix} \frac{1}{2}(m_{111} + m_{111} - m_{111}) & \frac{1}{2}(m_{122} + m_{122} - m_{221}) \\ \frac{1}{2}(m_{211} + m_{211} - m_{112}) & \frac{1}{2}(m_{222} + m_{222} - m_{222}) \end{bmatrix}\begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix}$$

$$+ \begin{bmatrix} m_{112} + m_{121} - m_{121} \\ m_{212} + m_{221} - m_{122} \end{bmatrix}\left[\dot{q}_1\dot{q}_2\right]$$

Christoffel Symbols

$$\frac{\partial m_{ij}}{\partial q_k}$$

$$b_{ijk} = \frac{1}{2}(m_{ijk} + m_{ikj} - m_{jki})$$

$$V = \begin{bmatrix} b_{111} & b_{122} \\ b_{211} & b_{222} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} 2b_{112} \\ 2b_{212} \end{bmatrix} [\dot{q}_1 \dot{q}_2]$$

$C(\mathbf{q})$ $B(\mathbf{q})$

$$\underset{(n\times n)}{C(\mathbf{q})}\underset{(n\times 1)}{[\dot{\mathbf{q}}^2]} = \begin{bmatrix} b_{1,11} & b_{1,22} & \cdots & b_{1,nn} \\ b_{2,11} & b_{2,22} & \cdots & b_{2,nn} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n,11} & b_{n,22} & \cdots & b_{n,nn} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \vdots \\ \dot{q}_n^2 \end{bmatrix}$$

$$\underset{(n\times\frac{(n-1)n}{2})}{B(\mathbf{q})} \underset{(\frac{(n-1)n}{2}\times 1)}{[\dot{\mathbf{q}}\dot{\mathbf{q}}]} = \begin{bmatrix} 2b_{1,12} & 2b_{1,13} & \cdots & 2b_{1,(n-1)n} \\ 2b_{2,12} & 2b_{2,13} & \cdots & 2b_{2,(n-1)n} \\ \vdots & \vdots & \vdots & \vdots \\ 2b_{n,12} & 2b_{n,13} & \cdots & 2b_{n,(n-1)n} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \dot{q}_2 \\ \dot{q}_1 \dot{q}_3 \\ \vdots \\ \dot{q}_{(n-1)} \dot{q}_n \end{bmatrix}$$

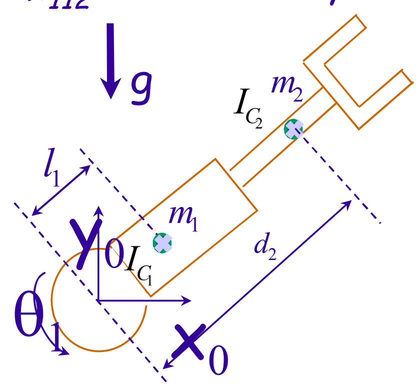
Centrifugal and Coriolis Vector V

$$b_{i,jk} = \frac{1}{2}\left(m_{ijk} + m_{ikj} - m_{jki}\right) \quad M = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix}$$

where $m_{ijk} = \dfrac{\partial m_{ij}}{\partial q_k}$; with $b_{iii} = 0$ and $b_{iji} = 0$ for $i > j$

For this manipulator, only m_{11} is configuration dependent - function of d_2. This implies that only m_{112} is non-zero,

$$m_{112} = 2 m_2 d_2 .$$



Matrix B

$$B = \begin{bmatrix} 2b_{112} \\ 0 \end{bmatrix} = \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix}.$$

Matrix C

$$C = \begin{bmatrix} 0 & b_{122} \\ b_{211} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix}.$$

Matrix V

$$V = \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix}\left[\dot{\theta}_1 \dot{d}_2\right] + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix}.$$

The Gravity Vector G

$$\mathbf{G} = -\left[J_{v_1}^T m_1 \mathbf{g} + J_{v_2}^T m_2 \mathbf{g}\right].$$

In frame {0}, ${}^0\mathbf{g} = \begin{pmatrix} 0 & -g & 0 \end{pmatrix}^T$ and the gravity vector is

$${}^0\mathbf{G} = -\begin{bmatrix} -l_1 s_1 & l_1 c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_1 g \\ 0 \end{bmatrix} - \begin{bmatrix} -d_2 s_1 & d_2 c_1 & 0 \\ c_1 & s_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_2 g \\ 0 \end{bmatrix}$$

and

$${}^0\mathbf{G} = \begin{bmatrix} \left(m_1 l_1 + m_2 d_2\right) g c_1 \\ m_2 g s_1 \end{bmatrix}$$

Equations of Motion

$$\begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{d}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{d}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix}$$

$$+ \begin{bmatrix} (m_1 l_1 + m_2 d_2) g c_1 \\ m_2 g s_1 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}.$$