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Nov 7, 2017

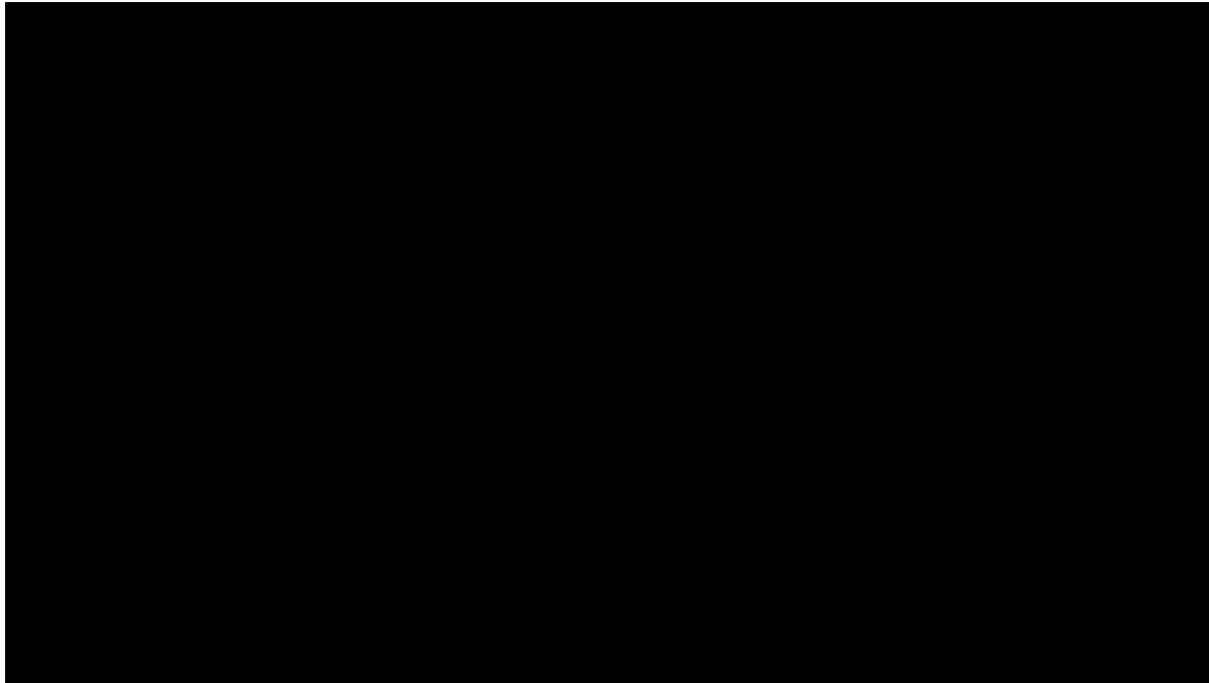
CMPUT 399

Intro Robotics & Mechatronics:

Statics & Dynamics

Some slides are taken from Prof. O. Khatib - lecture notes (Introduction to Robotics)
Some slides are taken from Prof. A. De Luca - lecture notes (Robotics 1)

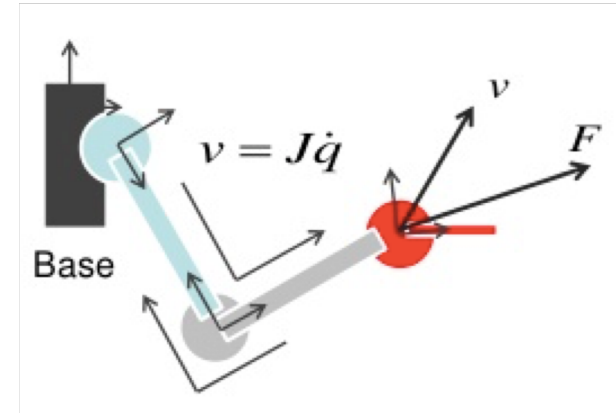
CASSIE biped robot



- CASSIE, a biped robot created by Agility Robotics (spin-off Oregon St. University)
 - Balance while walking
 - Interact with unknown environment
 - uses **sensors** embedded in its **legs** to keep its balance.
- Robots move from industry to **human environments** which is
 - **Unknown** (time varying) with lots of uncertainties
- Interaction with human -> **safety** -> **control**
 - **Statics** (force balance) >> **Dynamics** (force & motion)

Kinematics (review)

- Kinematics: $\dot{x} = J \dot{q}$
(relation between q , x)



- Inverse Kinematics:

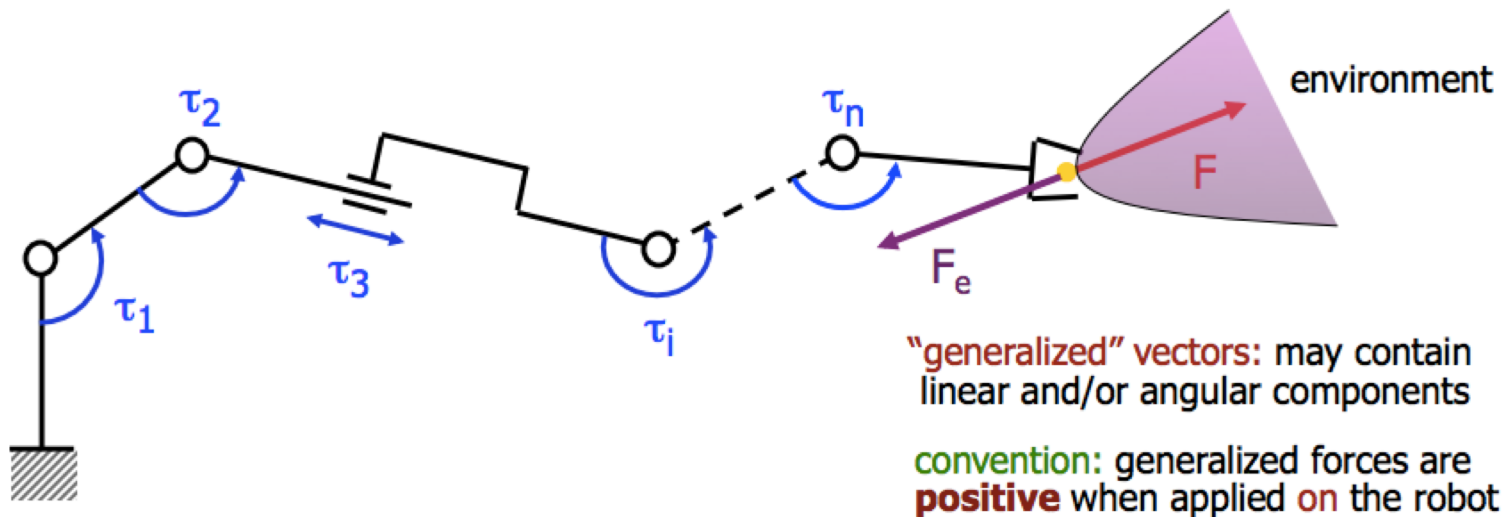
when J square and
non-singular



$$\dot{q} = J^{-1}(q) \dot{x}$$

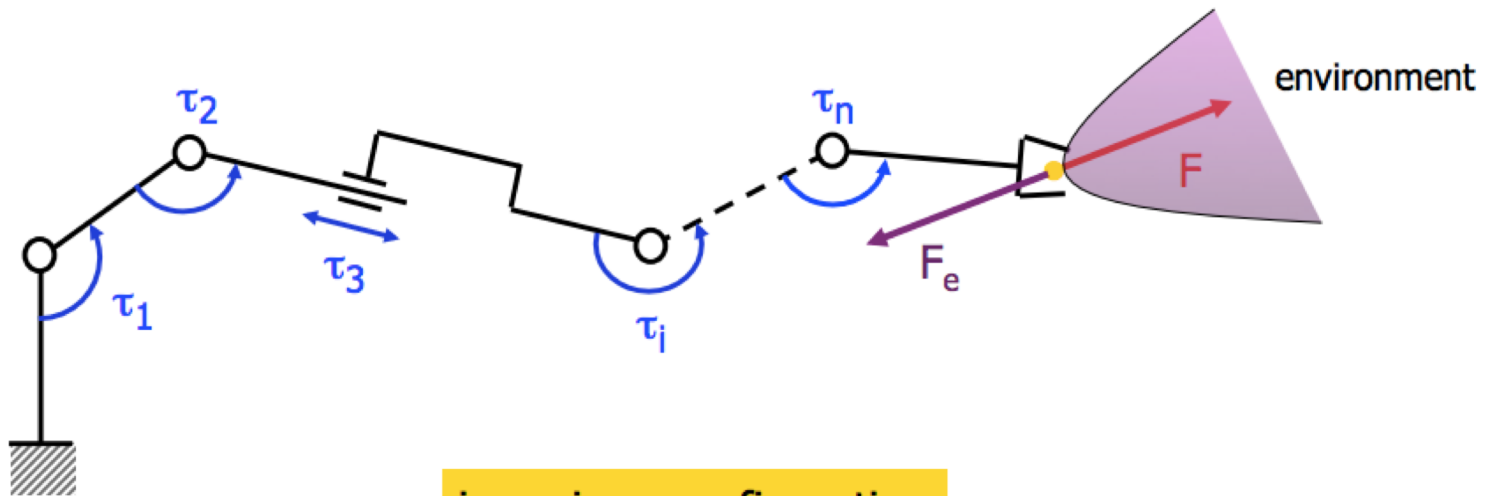
- near **singularity** of the Jacobian matrix (high $\dot{q}$)!
- for redundant robots ($n \neq m$), no standard “inverse” of a rectangular matrix $\rightarrow$ pseudo-inverse.

Generalized Forces and Torques



- τ = forces/torques exerted **by the motors** at the robot joints
- F = **equivalent** forces/torques exerted at the robot end-effector
- F_e = forces/torques exerted **by the environment** at the end-effector
- principle of action and reaction: $F_e = -F$
*reaction from environment is **equal and opposite** to the robot action on it*

Statics - Transformation of Forces

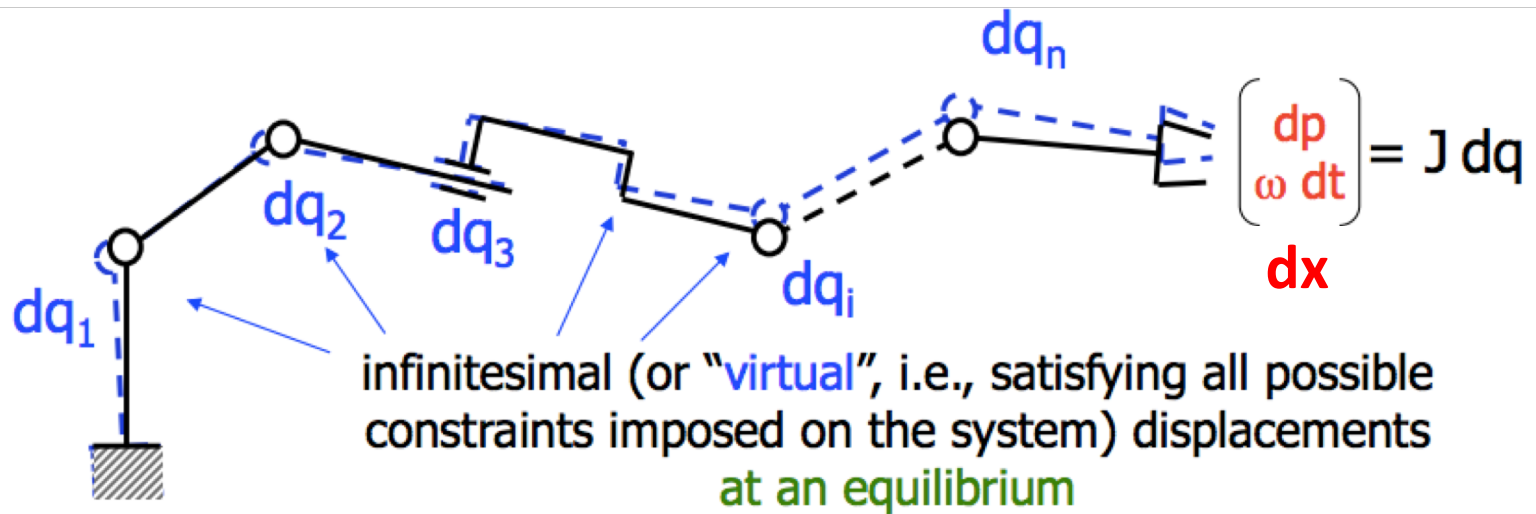


in a given configuration

- what is the transformation between F at robot end-effector and τ at joints?
- in **static equilibrium** conditions (i.e., **no motion**):
- what F will be exerted on environment by a τ applied at the robot joints?
 - what τ at the joints will balance a $F_e (= -F)$ exerted by the environment?

all equivalent formulations

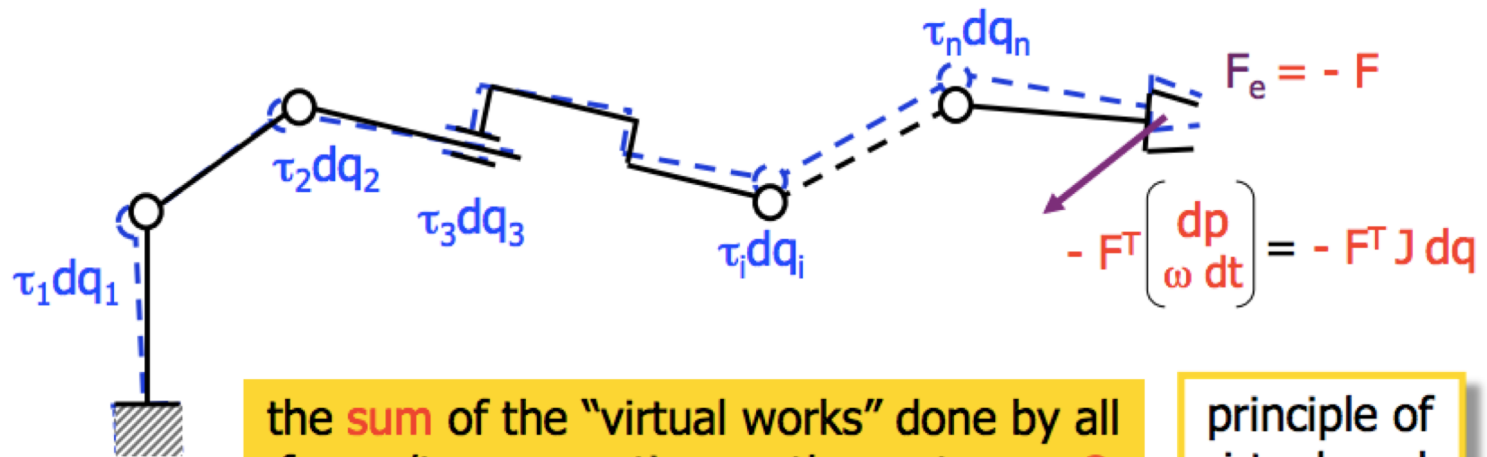
Virtual displacement and works



- without kinetic energy variation (zero acceleration)
- without dissipative effects (zero velocity)

the "virtual work" is the work done by all forces/torques acting **on** the system for a given virtual displacement

Principle of Virtual Work



the **sum** of the “virtual works” done by all forces/torques acting **on** the system = **0**

principle of virtual work

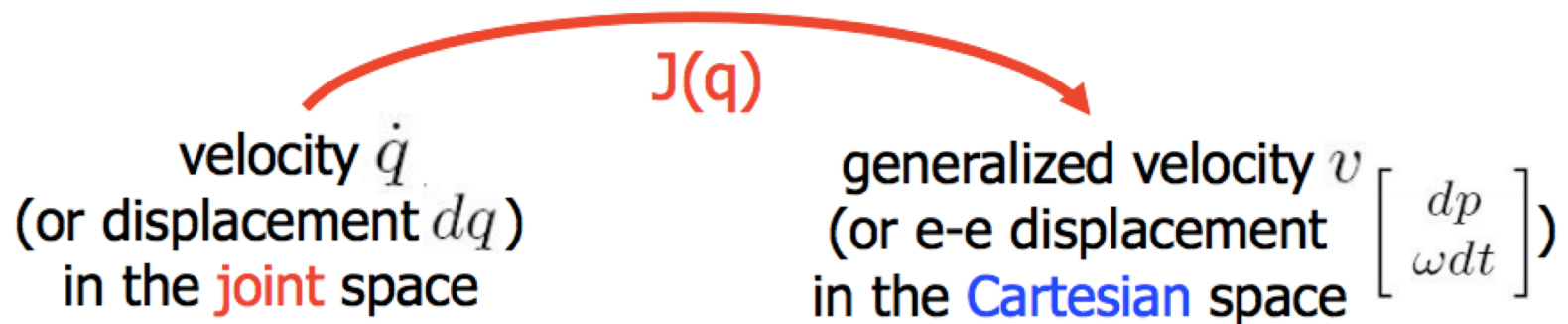
$$\tau^T dq - F^T \begin{bmatrix} dp \\ \omega dt \end{bmatrix} = \tau^T dq - F^T J dq = 0 \quad \boxed{\forall dq}$$



$$\tau = J^T(q)F$$

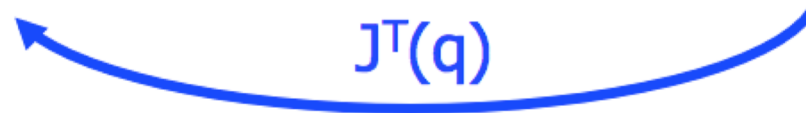
At **static equilibrium**, the virtual work done by active forces is **zero**.

Duality between Velocity & Force



forces/torques τ
at the **joints**

generalized forces F
at the **Cartesian** e-e

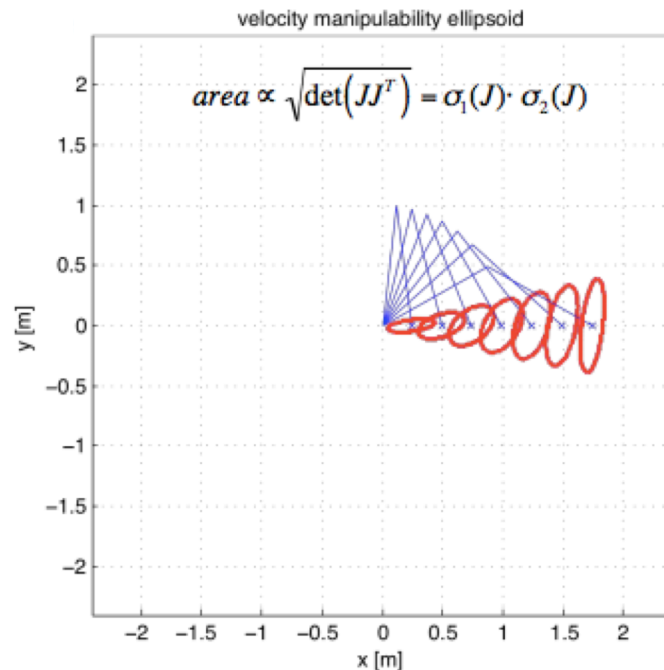


the singular configurations
for the **velocity map** are the **same**
as those for the **force map**

$$\rho(J) = \rho(J^T)$$

Duality between Velocity & Force

planar 2R arm with unitary links

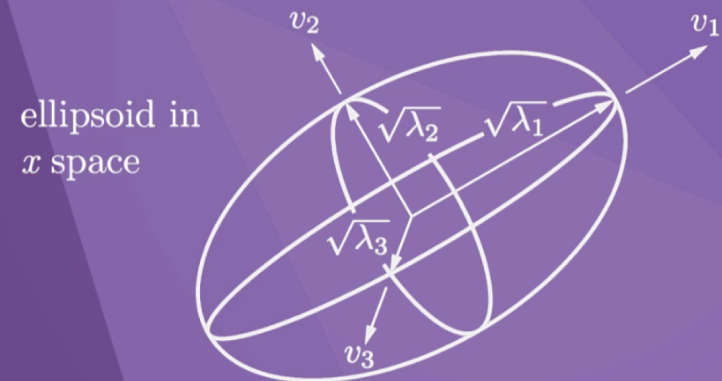


$$x^T A^{-1} x = 1$$

$A \in \mathbb{R}^{m \times m}$ (symmetric, positive definite)

e-vals of $A = \lambda_1, \dots, \lambda_m$

e-vecs of $A = v_1, \dots, v_m$



$$\dot{q}^T \dot{q} = 1 \quad \rightarrow \quad v^T J^\#^T J^\# v = 1$$

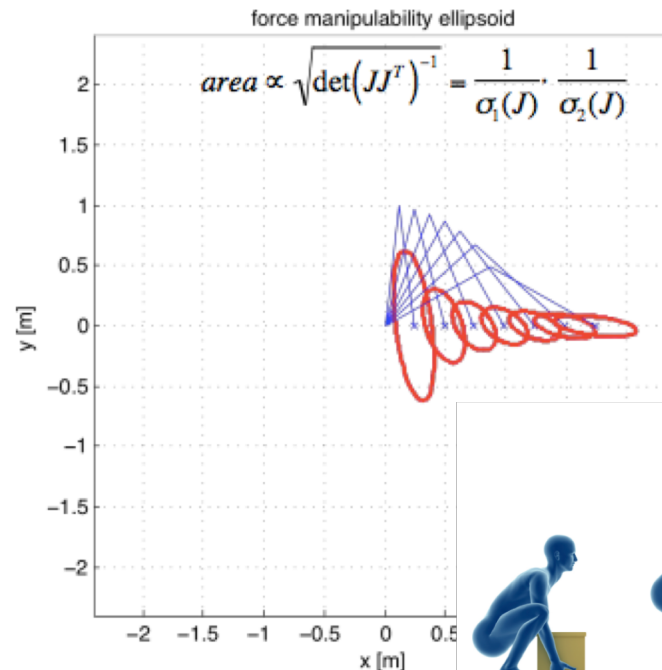
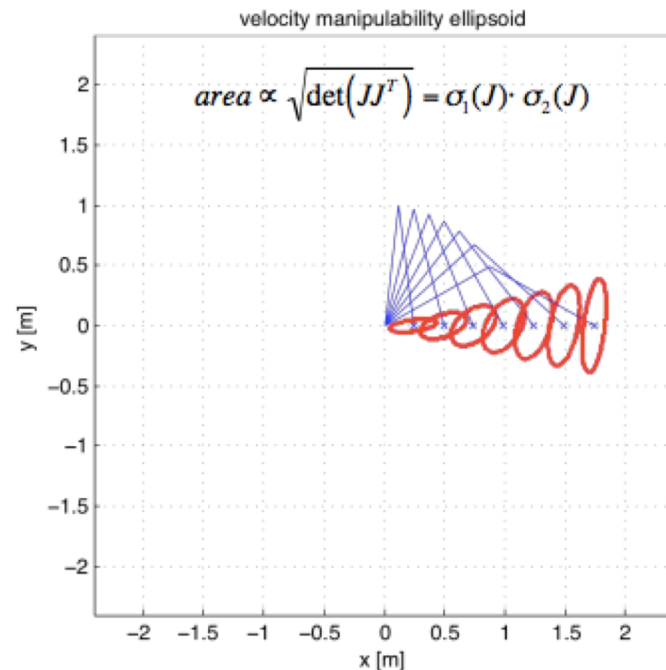
task velocity manipulability ellipsoid

$(JJ^T)^{-1}$ if $\rho = m$

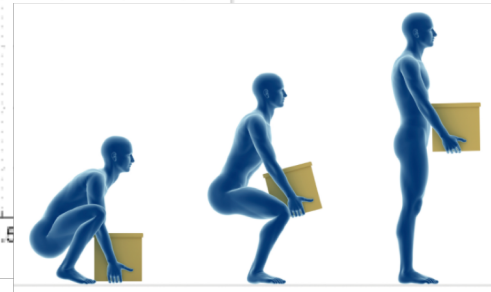
“how easily” can the end-effector be moved in the various directions of the task space

Duality between Velocity & Force

planar 2R arm with unitary links



@ singularity
robot cannot
move in certain
direction but
can sustain a lot
of forces (by the
structure of
mechanism)



$$\dot{q}^T \dot{q} = 1 \quad \Rightarrow \quad v^T J^{\#T} J^{\#} v = 1$$

task **velocity**
manipulability **ellipsoid**

$(JJ^T)^{-1}$ if $\rho = m$

“how easily” can the end-effector be
moved in the various directions of the
task space

$$\tau^T \tau = 1 \quad \Rightarrow \quad F^T J J^T F = 1$$

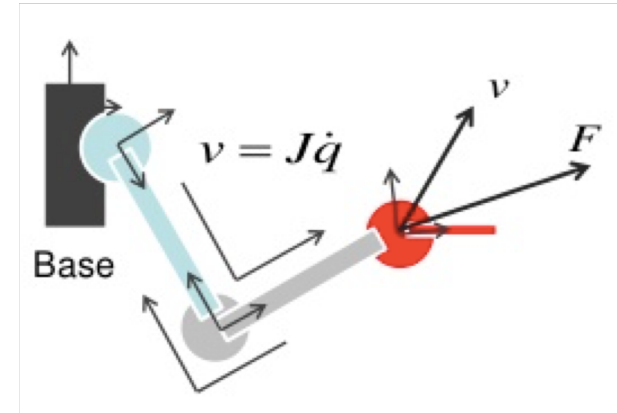
same directions of the principal
axes of the velocity ellipsoid, but
with semi-axes of **inverse** lengths

task **force**
manipulability **ellipsoid**

“how easily” can the end-effector apply **forces**
(or balance applied ones) in the various
directions of the task space

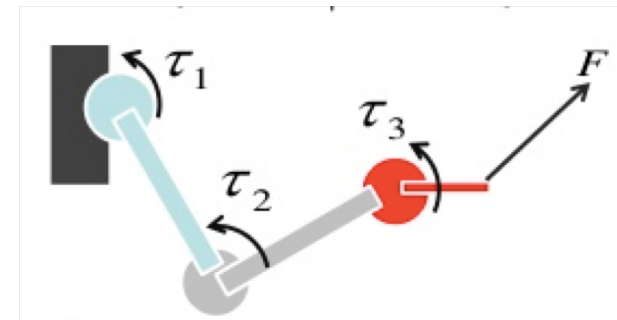
Kinematics - Statics **Duality**

- Kinematics: $\dot{x} = J \dot{q}$
(relation between $\mathbf{q}$, $\mathbf{x}$)



- Statics:
(relation between $\boldsymbol{\tau}$ and $\mathbf{F}$)

$$\boldsymbol{\tau} = J^T \mathbf{F}$$

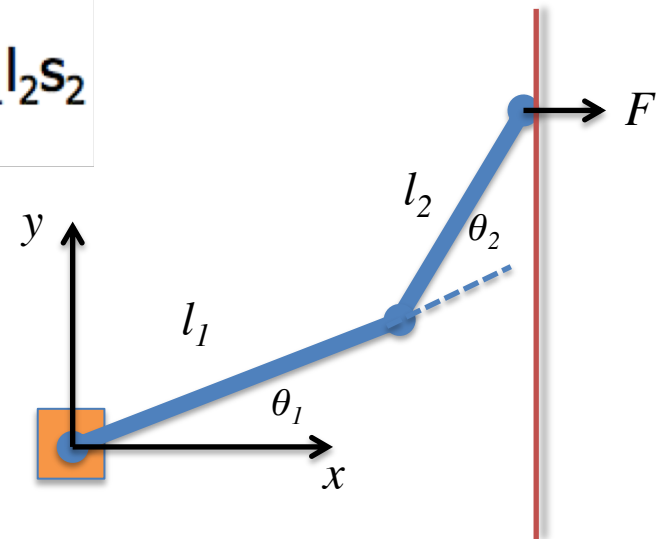


Example 1

- What are the joint torques required for the 2-link robot to push against the wall in x-direction?

$$J(q) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} \quad \det J(q) = l_1 l_2 s_2$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = J^T \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



Example 2: WAM robot video

- WAM robot: applying force/moment in all 6-dofs



$$\tau = \tau_g + J_{lin}^T F_{base} + J_{rot}^T M_{base}$$

$$\dot{x} = \begin{bmatrix} J_{lin} \\ - \\ J_{rot} \end{bmatrix} \dot{q}$$



$$F_x \leftarrow$$

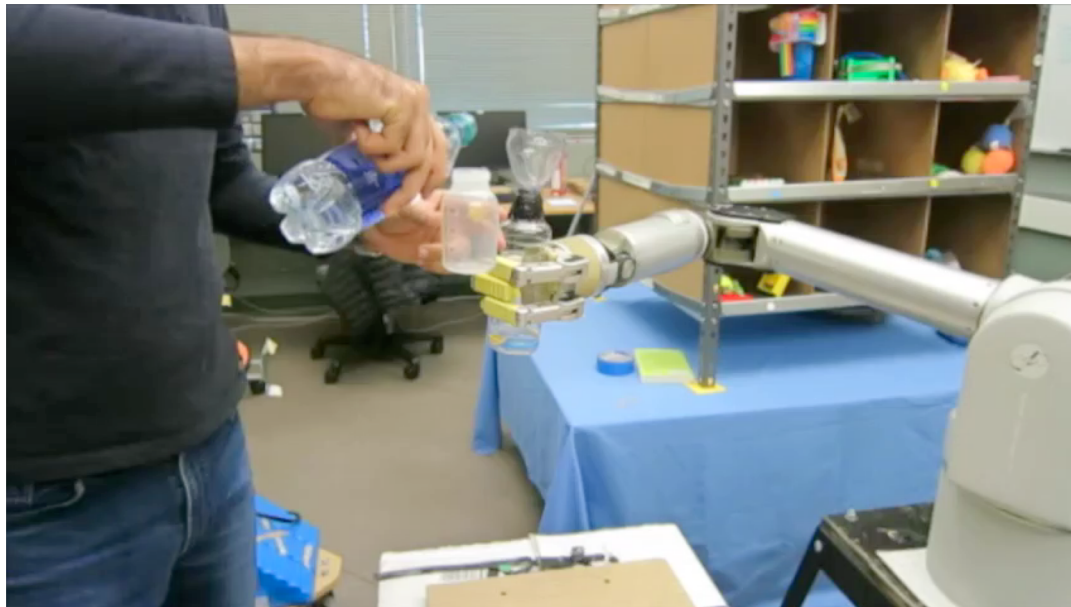
Example 3: estimation of external forces from joint torques measurement

- In static condition:

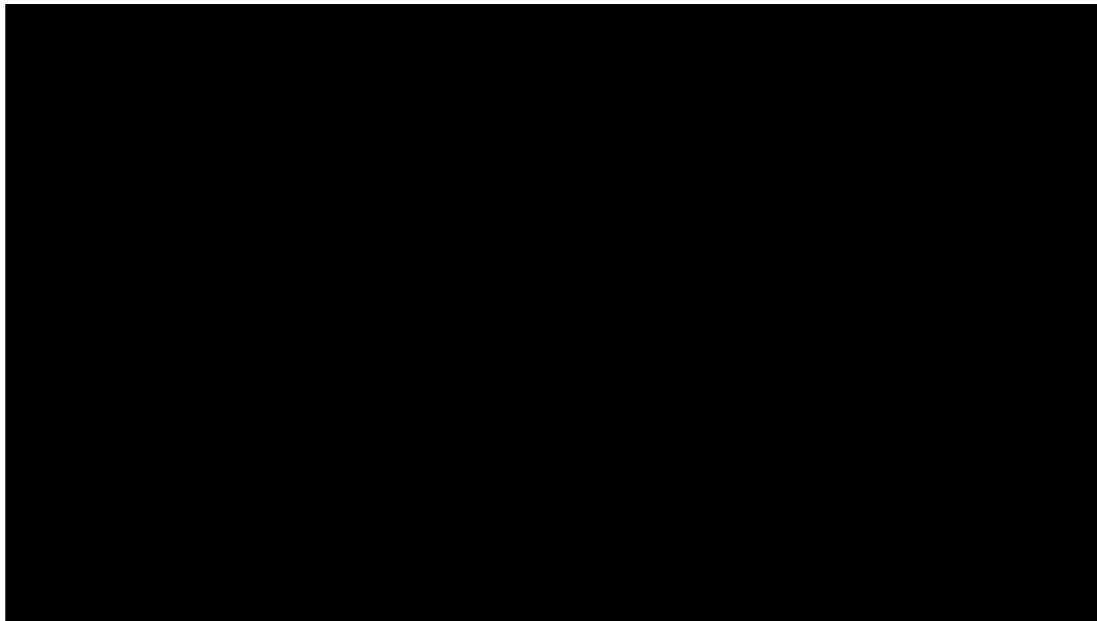
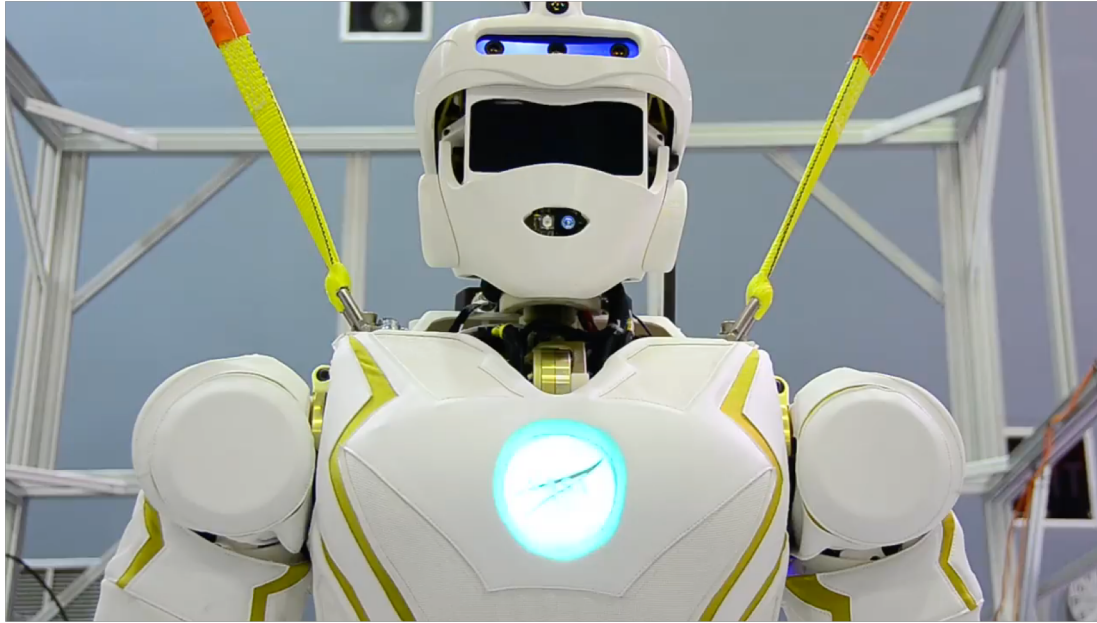
$$J_c^T F_c = \underbrace{\tau_{wam}}_{\text{wam.getJpintTorque()}} - \underbrace{\tau_g}_{\text{bt.calgrav_eval}}$$

$$F_c = J_c^\dagger (\tau_{wam} - \tau_g)$$

$$J_c^\dagger = (J_c J_c^T)^{-1} J_c$$



Valkyrie NASA robot



Dynamics

- Dynamics: relates **forces** and **acceleration**
 - Need to understand rigid body dynamics

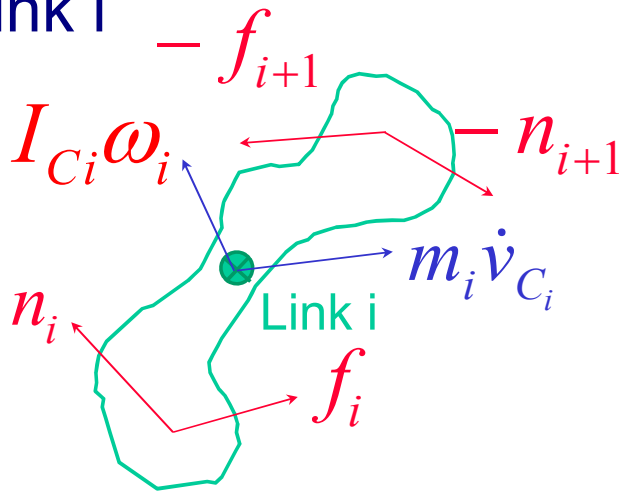
$$M(q)\ddot{q} + N(q, \dot{q})\dot{q} + g(q) = \tau$$

- **Newton – Euler method**
- **Lagrange formulation (energy-based)**

Rigid body dynamics (linear + angular)

Dynamic forces on Link i

$$I_{Ci} \dot{\omega}_i + \omega_i \times I_{Ci} \omega_i$$

$$m_i \dot{v}_{C_i} = \sum \text{forces}$$


$$I_{Ci} \dot{\omega}_i + \omega_i \times I_{Ci} \omega_i = \sum \text{moments} / c_i$$

Inertial forces/moments

$$F_i = m_i \dot{v}_{C_i}$$

Linear motion: (Newton Eqn)

$$N_i = I_{Ci} \dot{\omega}_i + \omega_i \times I_{Ci} \omega_i$$

Angular motion: (Euler Eqn)

Lagrange formulation (energy based)

Lagrange Equations

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = \tau$$

Lagrangian

L

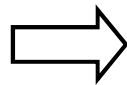
$= K$

$- U$

Kinetic Energy

Potential Energy

Since $U = U(q)$



$$\frac{d}{dt} \left(\frac{\partial K}{\partial \dot{q}} \right) - \frac{\partial K}{\partial q} + \frac{\partial U}{\partial q} = \tau$$

Inertial forces

Gravity vector

Inertial forces

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} = \tau - G \quad K = \frac{1}{2} \dot{q}^T M(q) \dot{q}$$

$$\frac{\partial K}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}} \left[\frac{1}{2} \dot{q}^T M(q) \dot{q} \right] = M(q) \dot{q}$$

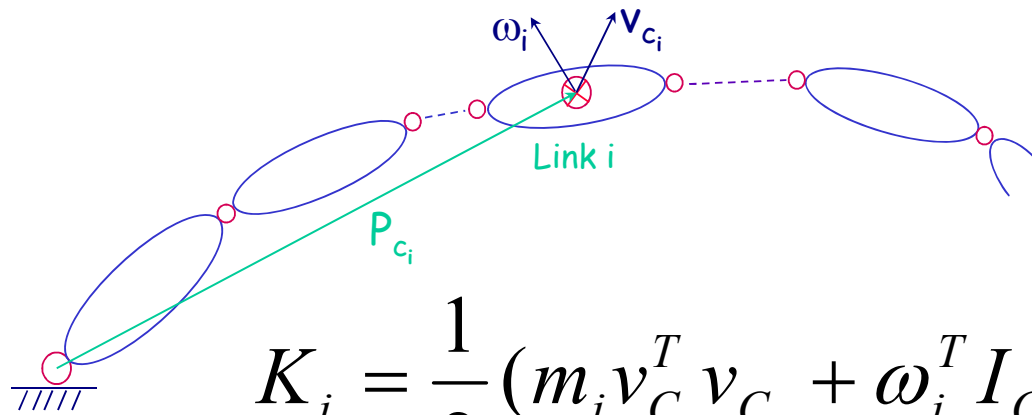
$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) = \frac{d}{dt}(M \dot{q}) = M \ddot{q} + \dot{M} \dot{q}$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}}\right) - \frac{\partial K}{\partial q} = M \ddot{q} + \dot{M} \dot{q} - \frac{1}{2} \begin{bmatrix} \dot{q}^T \frac{\partial M}{\partial q_1} \dot{q} \\ \vdots \\ \dot{q}^T \frac{\partial M}{\partial q_n} \dot{q} \end{bmatrix} = M \ddot{q} + V(q, \dot{q})$$

Kinetic energy of a link

Equations of Motion

Explicit Form

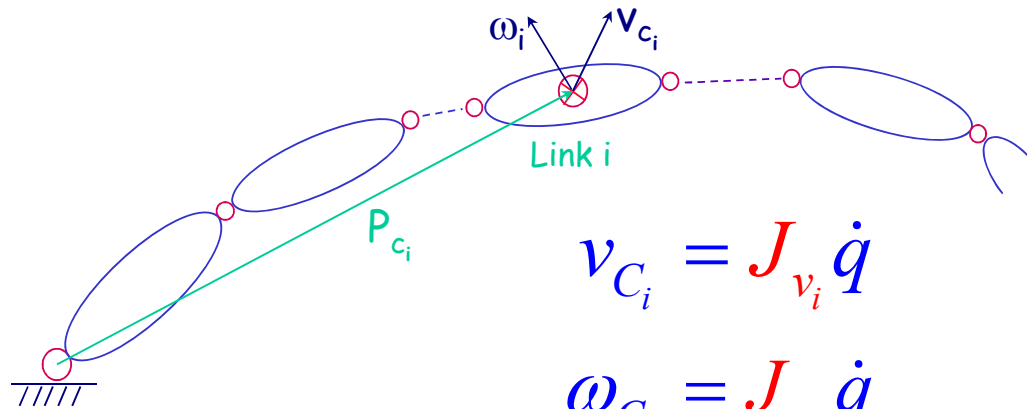


$$K_i = \frac{1}{2} (m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{C_i} \omega_i)$$

Total Kinetic Energy $\Rightarrow K = \sum_{i=1}^n K_i$

Equations of Motion

Explicit Form



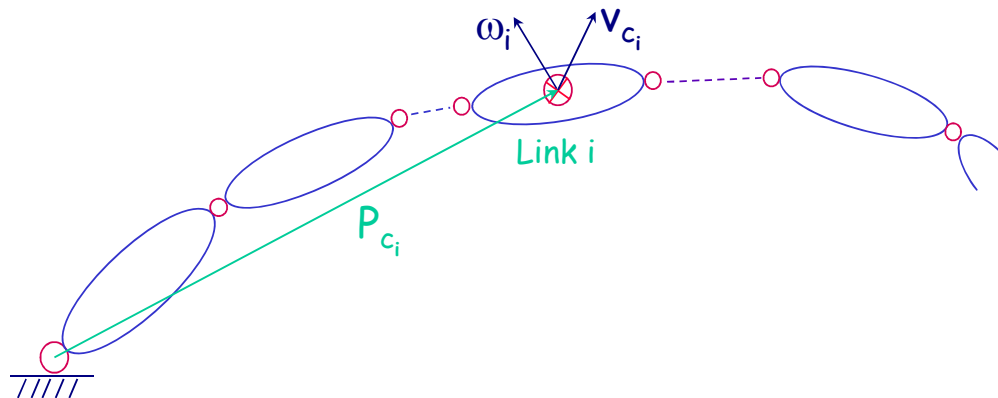
$$v_{C_i} = J_{v_i} \dot{q}$$

$$\omega_{C_i} = J_{\omega_i} \dot{q}$$

$$\begin{aligned} \frac{1}{2} \dot{q}^T M \dot{q} &= \frac{1}{2} \sum_{i=1}^n (m_i v_{C_i}^T v_{C_i} + \omega_i^T I_{C_i} \omega_i) \\ &= \frac{1}{2} \sum_{i=1}^n (m_i \dot{q}^T J_{v_i}^T J_{v_i} \dot{q} + \dot{q}^T J_{\omega_i}^T I_{C_i} J_{\omega_i} \dot{q}) \end{aligned}$$

Equations of Motion

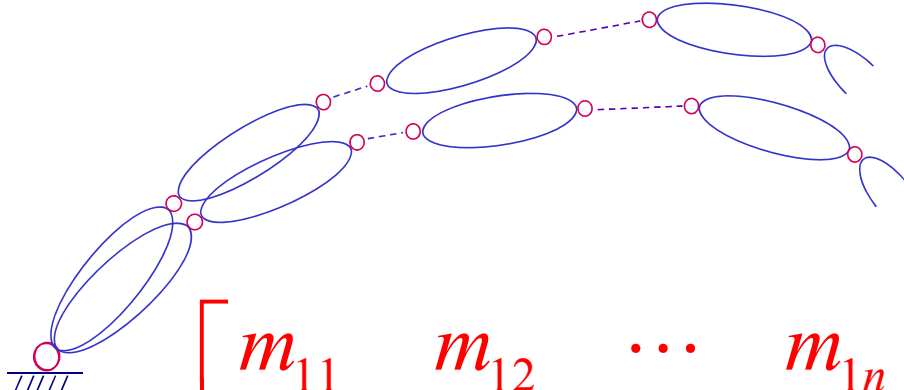
Explicit Form



$$\frac{1}{2} \dot{q}^T M \dot{q} = \frac{1}{2} \dot{q}^T \left[\sum_{i=1}^n (m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{C_i} J_{\omega_i}) \right] \dot{q}$$

$$M = \sum_{i=1}^n (m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{C_i} J_{\omega_i})$$

Properties of $M(q)$ matrix



$$M(q) = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}$$

$(n \times n)$

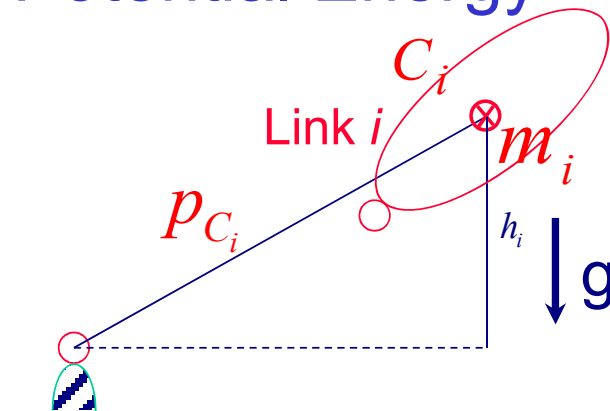
$M(q)$ is Symmetric:

$M(q)$ is Positive definite:

$$M = M^T \quad \text{i.e. } m_{12} = m_{21}, \dots$$

$$\xi^T M(q) \xi > 0 \quad \text{for all } \xi \neq 0$$

Potential Energy



$$U_i = m_i g_0 h_i + U_0$$

$$U_i = m_i (-g^T p_{C_i}) + U_0$$

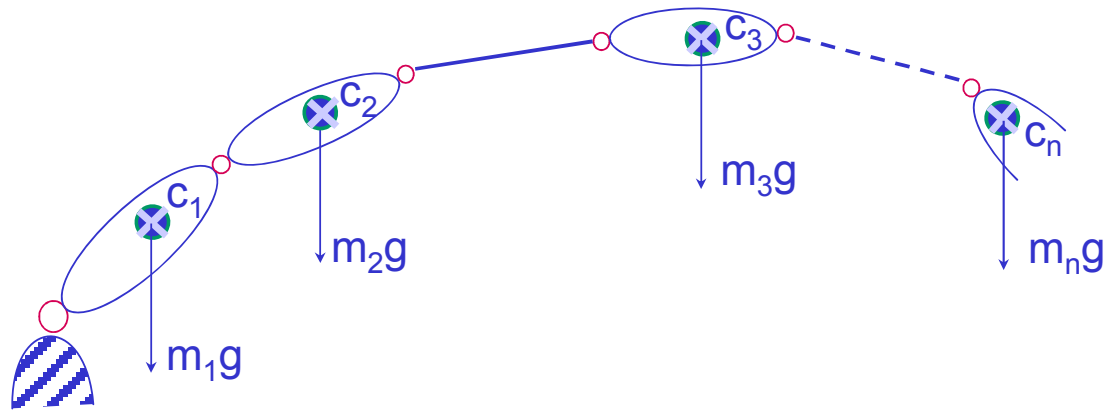
$$U = \sum U_i$$

Gravity Vector

$$G_j = \frac{\partial U}{\partial q_j} = - \sum_{i=1}^n (m_i g^T \frac{\partial \mathbf{p}_{C_i}}{\partial q_j})$$

$$G = - \begin{pmatrix} J_{v_1}^T & J_{v_2}^T & \dots & J_{v_n}^T \end{pmatrix} \begin{pmatrix} m_1 g \\ m_2 g \\ \vdots \\ m_n g \end{pmatrix}$$

Gravity Vector



$$G = -(J_{v_1}^T(m_1g) + J_{v_2}^T(m_2g) + \dots + J_{v_n}^T(m_ng))$$

Example: 1-Rev. Joint + 1-prismatic

taken from Prof. O. Khatib lecture notes

(CS223A - Introduction to Robotics)

<https://see.stanford.edu/Course/CS223A>

Matrix M

$$M = m_1 J_{v_1}^T J_{v_1} + J_{\omega_1}^T I_{C_1} J_{\omega_1} + m_2 J_{v_2}^T J_{v_2} + J_{\omega_2}^T I_{C_2} J_{\omega_2}$$

J_{v_1} and J_{v_2} : direct differentiation of the vectors:

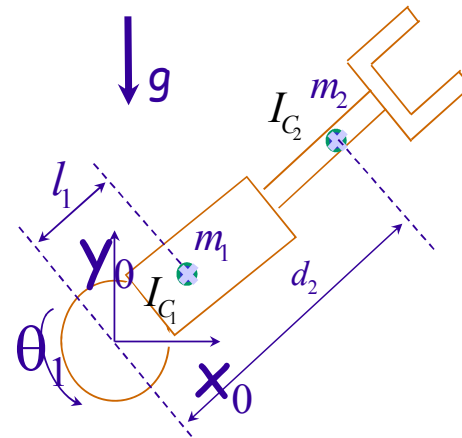
$${}^0\mathbf{p}_{C_1} = \begin{bmatrix} l_1 c_1 \\ l_1 s_1 \\ 0 \end{bmatrix}; \text{ and } {}^0\mathbf{p}_{C_2} = \begin{bmatrix} d_2 c_1 \\ d_2 s_1 \\ 0 \end{bmatrix}$$

In frame $\{0\}$, these matrices are:

$${}^0J_{v_1} = \begin{bmatrix} -l_1 s_1 & 0 \\ l_1 c_1 & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } {}^0J_{v_2} = \begin{bmatrix} -d_2 s_1 & c_1 \\ d_2 c_1 & s_1 \\ 0 & 0 \end{bmatrix}$$

This yields

$$m_1 ({}^0J_{v_1}^T {}^0J_{v_1}) = \begin{bmatrix} m_1 l_1^2 & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } m_2 ({}^0J_{v_2}^T {}^0J_{v_2}) = \begin{bmatrix} m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix}$$



The matrices J_{ω_1} and J_{ω_2} are given by

$$J_{\omega_1} = [\bar{\epsilon}_1 \quad \mathbf{z}_1 \quad 0] = \text{ and } J_{\omega_2} = [\bar{\epsilon}_1 \quad \mathbf{z}_1 \quad \bar{\epsilon}_2 \quad \mathbf{z}_2]$$

Joint 1 is revolute and joint 2 is prismatic:

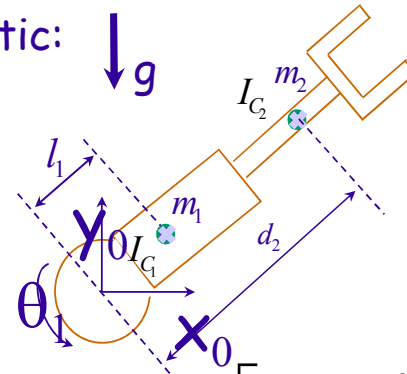
$${}^1J_{\omega_1} = {}^1J_{\omega_2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

And

$$({}^1J_{\omega_1}^T I_{C_1} {}^1J_{\omega_1}) = \begin{bmatrix} I_{zz1} & 0 \\ 0 & 0 \end{bmatrix}; \text{ and } ({}^1J_{\omega_2}^T I_{C_2} {}^1J_{\omega_2}) = \begin{bmatrix} I_{zz2} & 0 \\ 0 & 0 \end{bmatrix}$$

Finally,

$$M = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix}$$



Vector $V(\mathbf{q}, \dot{\mathbf{q}})$ $\frac{\partial M}{\partial \dot{q}_1}$

$$V = \dot{M}\dot{\mathbf{q}} - \frac{1}{2} \begin{bmatrix} \dot{\mathbf{q}}^T M_{q_1} \dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T M_{q_2} \dot{\mathbf{q}} \end{bmatrix} = \begin{pmatrix} \dot{m}_{11} & \dot{m}_{12} \\ \dot{m}_{12} & \dot{m}_{22} \end{pmatrix} \dot{\mathbf{q}} - \frac{1}{2} \begin{bmatrix} \dot{\mathbf{q}}^T \begin{pmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{pmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T \begin{pmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{pmatrix} \dot{\mathbf{q}} \end{bmatrix}$$

$$\dot{m}_{ij} = m_{ij1}\dot{q}_1 + m_{ij2}\dot{q}_2$$

$$V(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} \frac{1}{2}(m_{111} + m_{111} - m_{111}) & \frac{1}{2}(m_{122} + m_{122} - m_{221}) \\ \frac{1}{2}(m_{211} + m_{211} - m_{112}) & \frac{1}{2}(m_{222} + m_{222} - m_{222}) \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \frac{\partial m_{22}}{\partial q_2}$$

$$+ \begin{bmatrix} m_{112} + m_{121} - m_{121} \\ m_{212} + m_{221} - m_{122} \end{bmatrix} [\dot{q}_1 \dot{q}_2]$$

Christoffel Symbols

$$b_{ijk} = \frac{1}{2} (m_{ijk} + m_{ikj} - m_{jki})$$

$\frac{\partial m_{ij}}{\partial q_k}$

$$V = \begin{bmatrix} b_{111} & b_{122} \\ b_{211} & b_{222} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} 2b_{112} \\ 2b_{212} \end{bmatrix} [\dot{q}_1 \dot{q}_2]$$

$C(\mathbf{q})$

$B(\mathbf{q})$

$$C(\mathbf{q})[\dot{\mathbf{q}}^2] = \begin{bmatrix} b_{1,11} & b_{1,22} & \cdots & b_{1,nn} \\ b_{2,11} & b_{2,22} & \cdots & b_{2,nn} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n,11} & b_{n,22} & \cdots & b_{n,nn} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \\ \vdots \\ \dot{q}_n^2 \end{bmatrix}$$

$(n \times n) \quad (n \times 1)$

$$B(\mathbf{q}) [\dot{\mathbf{q}}\dot{\mathbf{q}}] = \begin{bmatrix} 2b_{1,12} & 2b_{1,13} & \cdots & 2b_{1,(n-1)n} \\ 2b_{2,12} & 2b_{2,13} & \cdots & 2b_{2,(n-1)n} \\ \vdots & \vdots & \vdots & \vdots \\ 2b_{n,12} & 2b_{n,13} & \cdots & 2b_{n,(n-1)n} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \dot{q}_2 \\ \dot{q}_1 \dot{q}_3 \\ \vdots \\ \dot{q}_{(n-1)} \dot{q}_n \end{bmatrix}$$

$(n \times \frac{(n-1)n}{2}) \quad (\frac{(n-1)n}{2} \times 1)$

Centrifugal and Coriolis Vector V

$$b_{i,jk} = \frac{1}{2}(m_{ijk} + m_{ikj} - m_{jki}) \quad M = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix}$$

where $m_{ijk} = \frac{\partial m_{ij}}{\partial q_k}$; with $b_{iii} = 0$ and $b_{iji} = 0$ for $i > j$

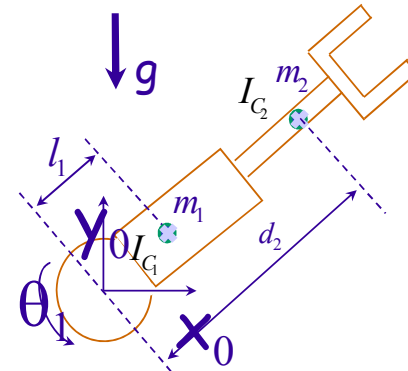
For this manipulator, only m_{11} is configuration dependent - function of d_2 . This implies that only m_{112} is non-zero,

$$m_{112} = 2m_2 d_2.$$

Matrix B $B = \begin{bmatrix} 2b_{112} \\ 0 \end{bmatrix} = \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix}.$

Matrix C $C = \begin{bmatrix} 0 & b_{122} \\ b_{211} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix}.$

Matrix V $V = \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{d}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix}.$



The Gravity Vector $\mathbf{G}$

$$\mathbf{G} = -[J_{v_1}^T m_1 \mathbf{g} + J_{v_2}^T m_2 \mathbf{g}].$$

In frame $\{0\}$, $\mathbf{g} = (0 \quad -g \quad 0)^T$ and the gravity vector is

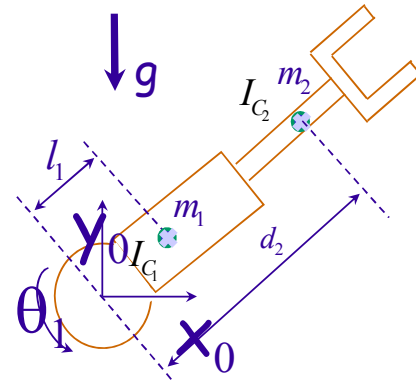
$${}^0\mathbf{G} = -\begin{bmatrix} -l_1 s_1 & l_1 c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_1 g \\ 0 \end{bmatrix} - \begin{bmatrix} -d_2 s_1 & d_2 c_1 & 0 \\ c_1 & s_1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_2 g \\ 0 \end{bmatrix}$$

and

$${}^0\mathbf{G} = \begin{bmatrix} (m_1 l_1 + m_2 d_2) g c_1 \\ m_2 g s_1 \end{bmatrix}$$

Equations of Motion

$$\begin{aligned}
 & \begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{d}_2 \end{bmatrix} \\
 & + \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \dot{d}_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix} \\
 & + \begin{bmatrix} (m_1 l_1 + m_2 d_2) g c_1 \\ m_2 g s_1 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}.
 \end{aligned}$$



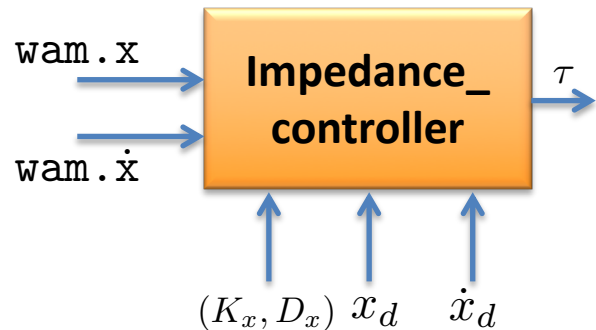
3. Impedance Controller Module (virtual spring)

- Impedance Controller:

$$M(q)\ddot{q} + N(q, \dot{q})\dot{q} + g(q) = \tau$$

$$\tau = \tau_g + D_x(\dot{x}_d - \dot{x}) + K_x(x_d - x)$$

$$\Rightarrow D_x(\dot{x}_d - \dot{x}) + K_x(x_d - x) = F$$



Stiffness K_x, K_y, K_z
Damping D_x, D_y, D_z

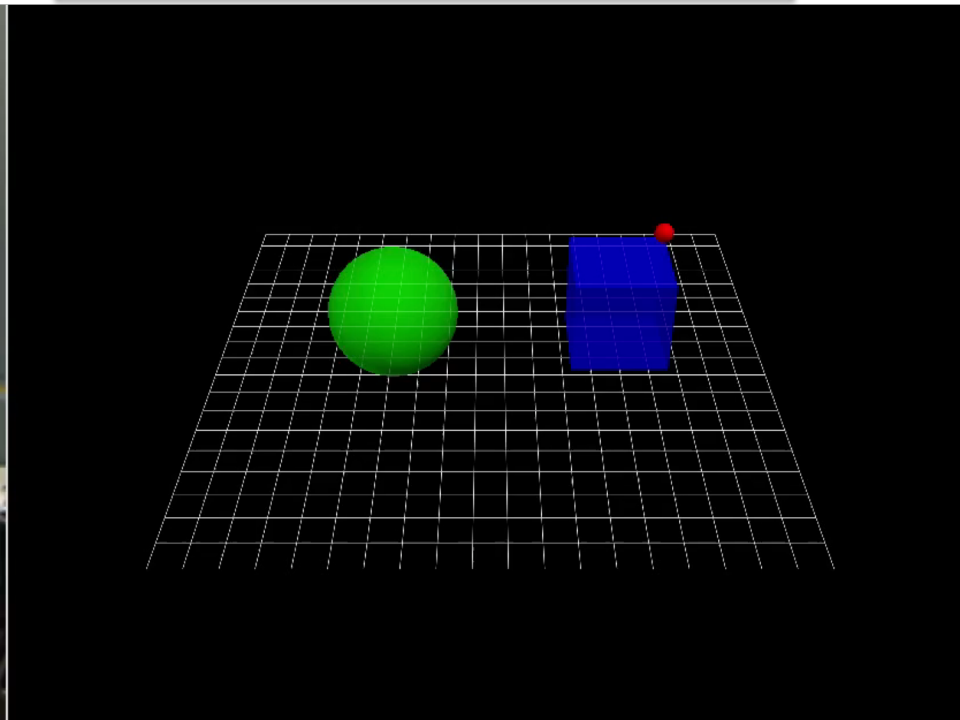
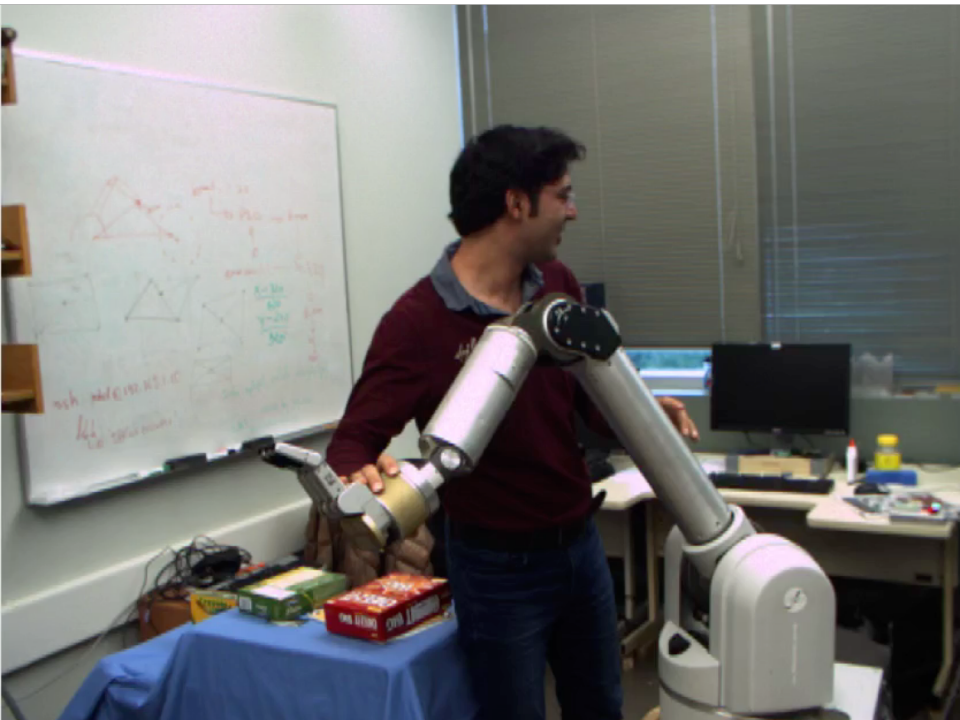
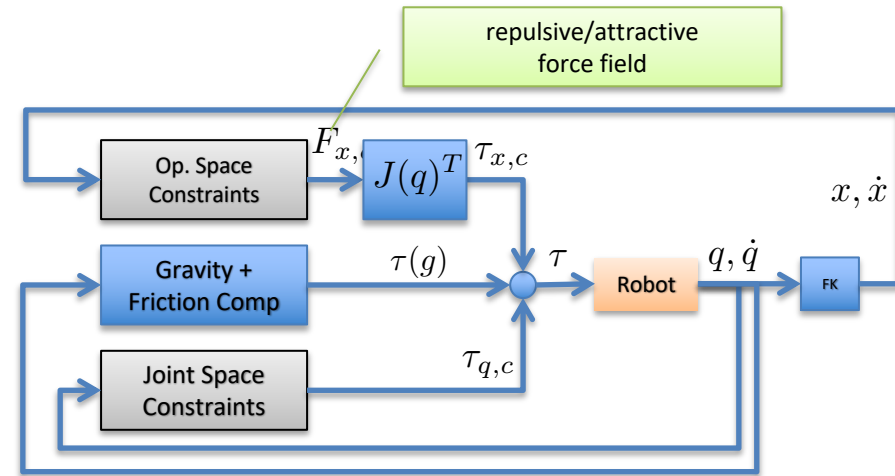


a. Virtual (Haptic) Fixtures

Virtual fixtures (VF): are software-generated force/position constraints.

VFs are obtained by adding **attractive/repulsive** force fields.

VFs improve manipulation by limiting movement into restricted regions (**Forbidden-region VF**) and/or influencing movement along desired paths (**Guidance VF**).



b. Bilateral tele-operation

Bilateral tele-op robotics is used for remote (space/under water) applications or for manipulation in dangerous environments.

Torque control + Virtual Fixtures improve the performance of bilateral tele-op robotics systems and enable them to execute complicated tasks. Attractive/repulsive force fields can be imposed on tele-op systems to avoid collision or detect contact with environment.

