Geometry and Homogeneous coordinoates

C340

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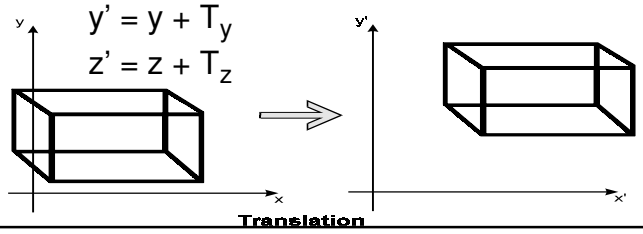
Basic transformations
Translation

- A translation is a straight line movement of an object from one postion to another.

  A point (x,y) is transformed to the point (x',y') by adding the translation distances T_x and T_y:

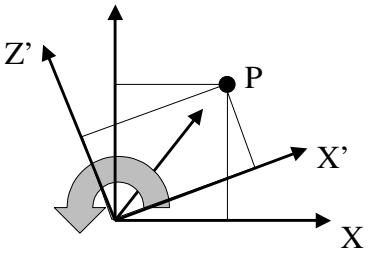
$$x' = x + T_x$$
$$y' = y + T_y$$
$$z' = z + T_z$$



Coordinate rotation

- Example: Around y-axis

$$\mathbf{p}' = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\nu & 0 & \sin\nu \\ 0 & 1 & 0 \\ -\sin\nu & 0 & \cos\nu \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{R}_y\mathbf{p}$$

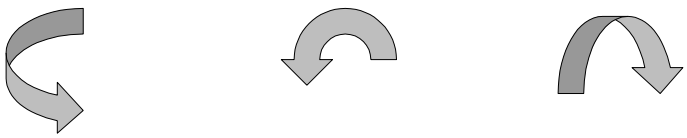


Euler angles

- Note: Successive rotations. Order matters.

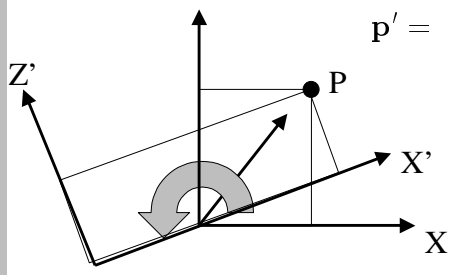
$$\mathbf{R} = \mathbf{R_z}\mathbf{R_y}\mathbf{R_x}$$

$$= \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\nu & 0 & \sin\nu \\ 0 & 1 & 0 \\ -\sin\nu & 0 & \cos\nu \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$



Rotation and translation

- Translation t' in new o' coordinates



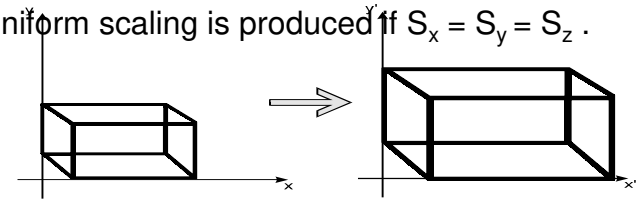
$$\mathbf{p}' = \begin{bmatrix} \cos\nu & 0 & \sin\nu \\ 0 & 1 & 0 \\ -\sin\nu & 0 & \cos\nu \end{bmatrix} \mathbf{p} + \mathbf{t}'$$

Basic transformations
Scaling

- A scaling transformation alters the scale of an object. Suppose a point (x,y) is transformed to the point (x',y') by a scaling with scaling factors S_x and S_y, then:

  $x' = x S_x$
  $y' = y S_y$
  $z' = z S$

- A uniform scaling is produced if $S_x = S_y = S_z$.



Scaling about the Origin

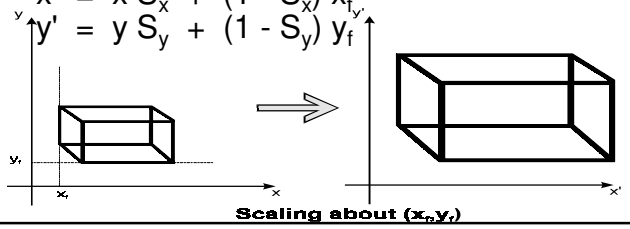
Basic transformations
Scaling

The previous scaling transformation leaves the origin unaltered. If the point (x_f,y_f) is to be the fixed point, the transformation is:

$$x' = x_f + (x - x_f)\, S_x$$
$$y' = y_f + (y - y_f)\, S_y$$

This can be rearranged to give:

$$x' = x\, S_x + (1 - S_x)\, x_f$$
$$y' = y\, S_y + (1 - S_y)\, y_f$$



Affine Geometric Transforms

In general, a point in n-D space transforms by

P' = rotate(point) + translate(point)

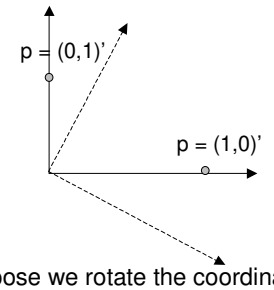
In 2-D space, this can be written as a matrix equation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} Cos(\theta) & -Sin(\theta) \\ Sin(\theta) & Cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} tx \\ ty \end{pmatrix}$$

In 3-D space (or n-D), this can generalized as a matrix equation:

$$p' = R\, p + T \quad \text{or} \quad p = R^t\, (p' - T)$$

A Simple 2-D Example



Suppose we rotate the coordinate system through 45 degrees (note that this is measured relative to the *rotated* system!

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} Cos(\pi/4) & -Sin(\pi/4) \\ Sin(\pi/4) & Cos(\pi/4) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} Cos(\pi/4) \\ Sin(\pi/4) \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} Cos(\pi/4) & -Sin(\pi/4) \\ Sin(\pi/4) & Cos(\pi/4) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -Sin(\pi/4) \\ Cos(\pi/4) \end{pmatrix}$$

Matrix representation and Homogeneous coordinates

- Often need to combine transformations to build the total transformation. If all transformations could be represented as matrix operations then the combination of transformations simply involves the multiplication of the respective matrices
- As 3D translations do not have a 3 x 3 matrix representation, we introduce homogeneous coordinates to allow a 4 x 4 matrix representation.

Homogeneous coordinates

- The Homogeneous coordinate corresponding to the point (x,y,z) is the triple (x_h, y_h, z_h, w) where:

$$x_h = wx$$
$$y_h = wy$$
$$z_h = wz$$

We can (initially) set $w = 1$.

- Suppose a point $P = (x,y,z,1)$ in the homogeneous coordinate system is mapped to a point $P' = (x',y',z',1)$ by a transformations, then the transformation can be expressed in matrix form.

Matrix representation and Homogeneous coordinates

- For the basic transformations we have:
  - Translation

$$P' = \begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

  - Scaling

$$P' = \begin{bmatrix} x' \\ y' \\ z' \\ w \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

Geometric Transforms

Using the idea of homogeneous transforms, we can write:

$$p' = \begin{pmatrix} & R & & T \\ 0 & 0 & 0 & 1 \end{pmatrix} p$$

R and T both require 3 parameters.

$$\mathrm{R} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\nu & 0 & \sin\nu \\ 0 & 1 & 0 \\ -\sin\nu & 0 & \cos\nu \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

Geometric Transforms

If we compute the matrix inverse, we find that

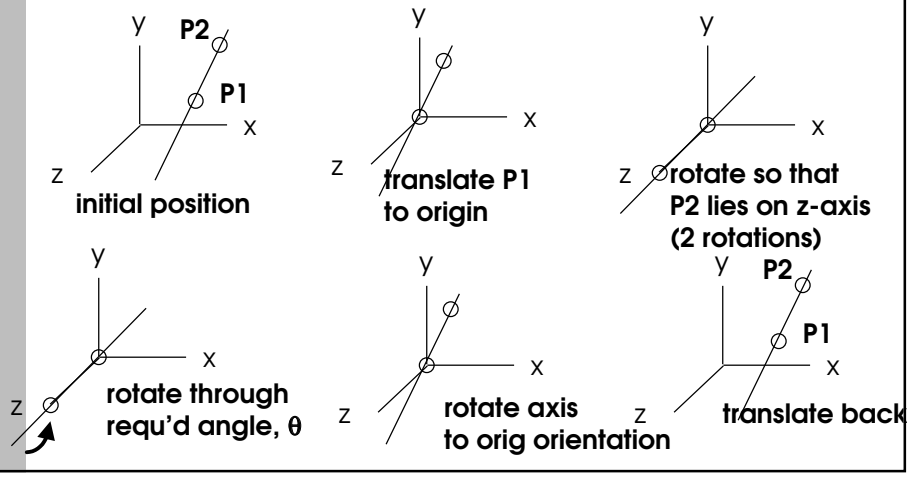
$$p = \begin{pmatrix} & R' & & -R'T \\ 0 & 0 & 0 & 1 \end{pmatrix} p'$$

R and T both require 3 parameters. These correspond to the 6 extrinsic parameters needed for camera calibration

Rotation about a Specified Axis

- It is useful to be able to rotate about any axis in 3D space
- This is achieved by composing 7 elementary transformations (next slide)

Rotation through θ about Specified Axis



Comparison:

- Homogeneous coordinates
  - Rotations and translations are represented in a uniform way
  - Successive transforms are composed using matrix products: y = Pn*..*P2*P1*x
- Affine coordinates
  - Non-uniform representations: y = Ax + b
  - Difficult to keep track of separate elements