Singularity and Nonsingularity

$n \times n$ matrix $\boldsymbol{A}$ is *nonsingular* if it has any of following equivalent properties:

1. Inverse of $\boldsymbol{A}$, denoted by $\boldsymbol{A}^{-1}$, exists

2. $\det(\boldsymbol{A}) \neq 0$

3. $\text{rank}(\boldsymbol{A}) = n$

4. For any vector $\boldsymbol{z} \neq \boldsymbol{o}$, $\boldsymbol{A}\boldsymbol{z} \neq \boldsymbol{o}$

Singularity and Nonsingularity, cont.

Solvability of $\boldsymbol{Ax} = \boldsymbol{b}$ depends on whether $\boldsymbol{A}$ is singular or nonsingular

If $\boldsymbol{A}$ is nonsingular, then $\boldsymbol{Ax} = \boldsymbol{b}$ has unique solution for any $\boldsymbol{b}$

If $\boldsymbol{A}$ is singular, then number of solutions is determined by $\boldsymbol{b}$

If $\boldsymbol{A}$ is singular and $\boldsymbol{Ax} = \boldsymbol{b}$, then $\boldsymbol{A}(\boldsymbol{x}+\gamma\boldsymbol{z}) = \boldsymbol{b}$ for any scalar γ, where $\boldsymbol{Az} = \boldsymbol{o}$ and $\boldsymbol{z} \neq \boldsymbol{o}$, so solution not unique

One solution:	nonsingular
No solution:	singular
∞ many solutions:	singular

Geometric Interpretation

In two dimensions, each equation determines straight line in plane

Intersection point of two lines is solution

If two straight lines not parallel (nonsingular), then intersection point unique

If two straight lines parallel (singular), then lines either do not intersect (no solution) or else coincide (any point along line is solution)

In higher dimensions, each equation determines hyperplane. If matrix nonsingular, intersection of hyperplanes is unique solution

Example: Nonsingularity

2×2 system

$$\begin{aligned} 2x_1 + 3x_2 &= b_1, \\ 5x_1 + 4x_2 &= b_2, \end{aligned}$$

or in matrix-vector notation

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \boldsymbol{b},$$

is nonsingular regardless of value of $\boldsymbol{b}$

For example, if $\boldsymbol{b} = [\,8 \quad 13\,]^T$, then $\boldsymbol{x} = [\,1 \quad 2\,]^T$ is unique solution

Example: Singularity

2×2 system

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \boldsymbol{b}$$

is singular regardless of value of $\boldsymbol{b}$

With $\boldsymbol{b} = [4 \quad 7]^T$, there is no solution

With $\boldsymbol{b} = [4 \quad 8]^T$, $\boldsymbol{x} = [\gamma \quad (4 - 2\gamma)/3]^T$ is solution for any real number γ