Vector Norms

Magnitude, modulus, or absolute value for scalars generalizes to *norm* for vectors

We will use only p-norms, defined by

$$\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

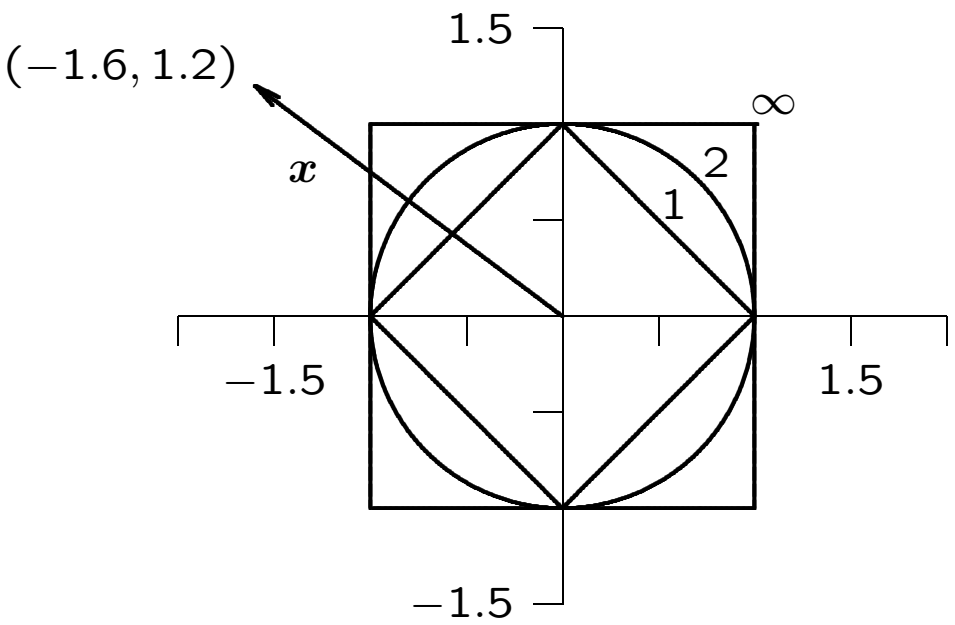
for integer $p > 0$ and n-vector $\boldsymbol{x}$

Important special cases:

- 1-norm: $\|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} |x_i|$
- 2-norm: $\|\boldsymbol{x}\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$
- ∞-norm: $\|\boldsymbol{x}\|_\infty = \max_i |x_i|$

Vector Norms, continued

Drawing shows unit sphere in two dimensions for each of these norms:



Norms have following values for vector shown:

$$\|\boldsymbol{x}\|_1 = 2.8, \quad \|\boldsymbol{x}\|_2 = 2.0, \quad \|\boldsymbol{x}\|_\infty = 1.6$$

In general, for any vector $\boldsymbol{x}$ in $\mathbb{R}^n$,

$$\|\boldsymbol{x}\|_1 \geq \|\boldsymbol{x}\|_2 \geq \|\boldsymbol{x}\|_\infty$$

Properties of Vector Norms

For any vector norm,

1. $\|\boldsymbol{x}\| > \boldsymbol{o}$ if $\boldsymbol{x} \neq \boldsymbol{o}$

2. $\|\gamma\boldsymbol{x}\| = |\gamma| \cdot \|\boldsymbol{x}\|$ for any scalar γ

3. $\|\boldsymbol{x}+\boldsymbol{y}\| \leq \|\boldsymbol{x}\|+\|\boldsymbol{y}\|$ (triangle inequality)

In more general treatment, these properties taken as *definition* of vector norm

Useful variation on triangle inequality:

$$| \, \|\boldsymbol{x}\| - \|\boldsymbol{y}\| \, | \leq \|\boldsymbol{x} - \boldsymbol{y}\|$$

Matrix Norms

Matrix norm corresponding to given vector norm defined by

$$\|\boldsymbol{A}\| = \max_{\boldsymbol{x} \neq \boldsymbol{o}} \frac{\|\boldsymbol{A}\boldsymbol{x}\|}{\|\boldsymbol{x}\|}$$

Norm of matrix measures maximum stretching matrix does to any vector in given vector norm

Matrix norm corresponding to vector 1-norm is maximum absolute column sum,

$$\|\boldsymbol{A}\|_1 = \max_j \sum_{i=1}^{n} |a_{ij}|$$

Matrix norm corresponding to vector ∞-norm is maximum absolute row sum,

$$\|\boldsymbol{A}\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

Properties of Matrix Norms

Any matrix norm satisfies:

1. $\|\boldsymbol{A}\| > \boldsymbol{O}$ if $\boldsymbol{A} \neq \boldsymbol{O}$

2. $\|\gamma \boldsymbol{A}\| = |\gamma| \cdot \|\boldsymbol{A}\|$ for any scalar γ

3. $\|\boldsymbol{A} + \boldsymbol{B}\| \leq \|\boldsymbol{A}\| + \|\boldsymbol{B}\|$

Matrix norms we have defined also satisfy

4. $\|\boldsymbol{A}\boldsymbol{B}\| \leq \|\boldsymbol{A}\| \cdot \|\boldsymbol{B}\|$

5. $\|\boldsymbol{A}\boldsymbol{x}\| \leq \|\boldsymbol{A}\| \cdot \|\boldsymbol{x}\|$ for any vector $\boldsymbol{x}$

Condition Number of Matrix

Condition number of square nonsingular matrix $\boldsymbol{A}$ defined by

$$\text{cond}(\boldsymbol{A}) = \|\boldsymbol{A}\| \cdot \|\boldsymbol{A}^{-1}\|$$

By convention, $\text{cond}(\boldsymbol{A}) = \infty$ if $\boldsymbol{A}$ singular

Since

$$\|\boldsymbol{A}\| \cdot \|\boldsymbol{A}^{-1}\| = \left(\max_{\boldsymbol{x}\neq\boldsymbol{o}} \frac{\|\boldsymbol{A}\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\right) \cdot \left(\min_{\boldsymbol{x}\neq\boldsymbol{o}} \frac{\|\boldsymbol{A}\boldsymbol{x}\|}{\|\boldsymbol{x}\|}\right)^{-1},$$

condition number measures ratio of maximum stretching to maximum shrinking matrix does to any nonzero vectors

Large $\text{cond}(\boldsymbol{A})$ means $\boldsymbol{A}$ nearly singular

Properties of Condition Number

1. For any matrix $\boldsymbol{A}$, $\text{cond}(\boldsymbol{A}) \geq 1$

2. For identity matrix, $\text{cond}(\boldsymbol{I}) = 1$

3. For any matrix $\boldsymbol{A}$ and scalar γ,

$$\text{cond}(\gamma \boldsymbol{A}) = \text{cond}(\boldsymbol{A})$$

4. For any diagonal matrix $\boldsymbol{D} = \text{diag}(d_i)$,

$$\text{cond}(\boldsymbol{D}) = (\max |d_i|)/(\min |d_i|)$$

Computing Condition Number

Definition of condition number involves matrix inverse, so nontrivial to compute

Computing condition number from definition would require much more work than computing solution whose accuracy to be assessed

In practice, condition number estimated inexpensively as byproduct of solution process

Matrix norm $\|\boldsymbol{A}\|$ easily computed as maximum absolute column sum (or row sum, depending on norm used)

Estimating $\|\boldsymbol{A}^{-1}\|$ at low cost more challenging

Computing Condition Number, cont.

From properties of norms, if $\boldsymbol{Az} = \boldsymbol{y}$, then

$$\frac{\|\boldsymbol{z}\|}{\|\boldsymbol{y}\|} \leq \|\boldsymbol{A}^{-1}\|,$$

and bound achieved for optimally chosen $\boldsymbol{y}$

Efficient condition estimators heuristically pick $\boldsymbol{y}$ with large ratio $\|\boldsymbol{z}\|/\|\boldsymbol{y}\|$, yielding good estimate for $\|\boldsymbol{A}^{-1}\|$

Good software packages for linear systems provide efficient and reliable condition estimator

Error Bounds

Condition number yields error bound for computed solution to linear system

Let $\boldsymbol{x}$ be solution to $\boldsymbol{Ax} = \boldsymbol{b}$, and let $\hat{\boldsymbol{x}}$ be solution to $\boldsymbol{A}\hat{\boldsymbol{x}} = \boldsymbol{b} + \Delta\boldsymbol{b}$

If $\Delta\boldsymbol{x} = \hat{\boldsymbol{x}} - \boldsymbol{x}$, then

$$\boldsymbol{b} + \Delta\boldsymbol{b} = \boldsymbol{A}(\hat{\boldsymbol{x}}) = \boldsymbol{A}(\boldsymbol{x} + \Delta\boldsymbol{x}) = \boldsymbol{Ax} + \boldsymbol{A}\Delta\boldsymbol{x},$$

which leads to bound

$$\frac{\|\Delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq \text{cond}(\boldsymbol{A})\frac{\|\Delta\boldsymbol{b}\|}{\|\boldsymbol{b}\|}$$

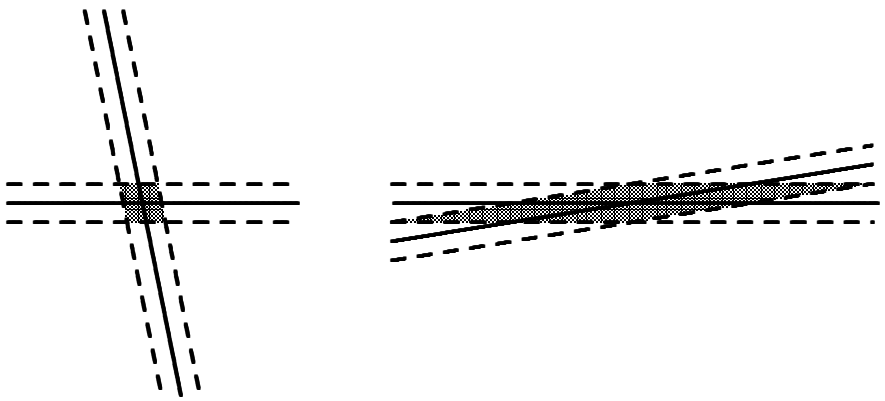
for possible relative change in solution due to relative change in right-hand side $\boldsymbol{b}$

Error Bounds, continued

Similar result holds for relative change in matrix: if $(\boldsymbol{A} + \boldsymbol{E})\hat{\boldsymbol{x}} = \boldsymbol{b}$, then

$$\frac{\|\Delta \boldsymbol{x}\|}{\|\hat{\boldsymbol{x}}\|} \leq \text{cond}(\boldsymbol{A}) \frac{\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|}$$

In two dimensions, uncertainty in intersection point of two lines depends on whether lines nearly parallel



well-conditioned ill-conditioned

Error Bounds, continued

If input data accurate to machine precision, then bound for relative error in computed solution given by

$$\frac{\|\widehat{\boldsymbol{x}} - \boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq \text{cond}(\boldsymbol{A})\, \epsilon_{\text{mach}}$$

Computed solution loses about $\log_{10}(\text{cond}(\boldsymbol{A}))$ decimal digits accuracy relative to accuracy of input

We will later see example using 3-digit precision for problem with cond $> 10^3$, which yields no correct digits in solution

Caveats

1. Normwise analysis bounds relative error in *largest* components of solution; relative error in smaller components can be much larger

Componentwise error bounds can be obtained, but somewhat more complicated

2. Conditioning of system affected by scaling

Ill-conditioning can result from poor scaling as well as near singularity

Rescaling can help former, but not latter

Residual

Residual vector of approximate solution $\widehat{\boldsymbol{x}}$ to linear system $\boldsymbol{Ax} = \boldsymbol{b}$ defined by

$$\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{A}\widehat{\boldsymbol{x}}$$

In theory, if $\boldsymbol{A}$ nonsingular, then $\|\widehat{\boldsymbol{x}} - \boldsymbol{x}\| = \boldsymbol{o}$ if, and only if, $\|\boldsymbol{r}\| = \boldsymbol{o}$, but not necessarily small simultaneously

Since

$$\frac{\|\Delta \boldsymbol{x}\|}{\|\widehat{\boldsymbol{x}}\|} \leq \text{cond}(\boldsymbol{A}) \frac{\|\boldsymbol{r}\|}{\|\boldsymbol{A}\| \cdot \|\widehat{\boldsymbol{x}}\|},$$

small relative residual implies small relative error only if $\boldsymbol{A}$ well-conditioned

Residual, continued

If computed solution $\hat{\boldsymbol{x}}$ exactly satisfies

$$(\boldsymbol{A} + \boldsymbol{E})\hat{\boldsymbol{x}} = \boldsymbol{b},$$

then

$$\frac{\|\boldsymbol{r}\|}{\|\boldsymbol{A}\| \ \|\hat{\boldsymbol{x}}\|} \leq \frac{\|\boldsymbol{E}\|}{\|\boldsymbol{A}\|},$$

so large *relative residual* implies large backward error in matrix, and algorithm used to compute solution is unstable

Stable algorithm yields small relative residual regardless how ill-conditioned nonsingular system may be

Solving Linear Systems

To solve linear system, transform it into one whose solution is same but easier to compute

What type of transformation of linear system leaves solution unchanged?

We can premultiply (from left) both sides of linear system $\boldsymbol{Ax} = \boldsymbol{b}$ by any nonsingular matrix $\boldsymbol{M}$ without affecting solution

Solution to $\boldsymbol{MAx} = \boldsymbol{Mb}$ is given by

$$\boldsymbol{x} = (\boldsymbol{MA})^{-1}\boldsymbol{Mb} = \boldsymbol{A}^{-1}\boldsymbol{M}^{-1}\boldsymbol{Mb} = \boldsymbol{A}^{-1}\boldsymbol{b}$$