Matrices & Vectors

Aims Of This Section

In this section we examine the solution of a linear system of equations.

- describe the basic Gaussian elimination method for computing a solution
- examine the LU decomposition method
- briefly look at matrix inversion
- return to the LU method and show how an improvement in accuracy may be obtained using iterative refinement
- look at some commercial Software Libraries
  - methods available
  - the use of special types of coefficient matrix to enhance efficiency
  - use of library routines

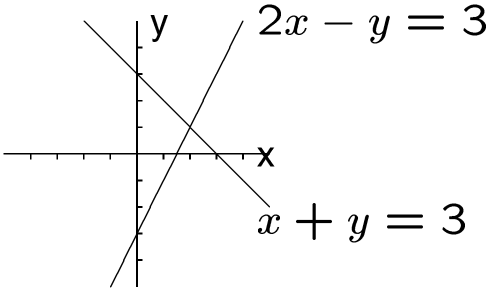
Revision, Definitions, Terminology

What is a matrix?

We may want to solve the simultaneous equations:

$$\begin{aligned} 2x - y &= 3 \\ x + y &= 3 \end{aligned}$$

These equations describe the locii of two straight lines on a plane, and the solution required is the point at which they intersect.



Solution: $x = 2, \quad y = 1$

If there are two unknowns (x and y above), then each equation defines a line on the (x, y) plane, and the solution is either a point (if the lines intersect at $x = x_0, y = y_0$), or no solution if the lines are parallel.

- matrices provide a shorthand notation for describing and solving sets of linear equations

- given a set of linear equations:

$$
\begin{array}{ccccc}
a_{11}x_1 & + \cdots + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + \cdots + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & \\
a_{n1}x_1 & + \cdots + & a_{nn}x_n & = & b_n
\end{array}
\tag{1}
$$

  we may rewrite above equations in the matrix notation:

$$
\mathbf{Ax} = \mathbf{b}
$$

  where **A** is called the *coefficient matrix*, **b** the *right-hand side vector*, and **x** the *solution vector* to be determined

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

Vectors

- a vector is a special case of a matrix in which the number of rows or columns is 1. For example,

$$\begin{bmatrix} -1 & 4 & 0 & 5 \end{bmatrix}$$

is a 1×3 matrix, or row vector

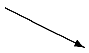
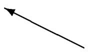
$$\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

is a 3×1 matrix, or column vector.

- vectors have both magnitude and direction

  for example:

 $\begin{bmatrix} 2 & 1 \end{bmatrix}$  $\begin{bmatrix} -2 & -1 \end{bmatrix}$

 $\begin{bmatrix} 2 & -1 \end{bmatrix}$  $\begin{bmatrix} -2 & 1 \end{bmatrix}$

- Note:
  - various notations are used for vectors, including **v**, $\vec{v}$, $\bar{v}$ and $\widehat{v}$.
  - in this section (and most applications), the term vector refers to a *column* vector.

Matrix Operations

- addition, subtraction
  - only appropriate for matrices of same size (ie. both $n \times m$)
  - if

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}$$

then

$$\begin{aligned} \mathbf{C} &= \mathbf{A} + \mathbf{B} \\ &= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1m} + b_{1m} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2m} + b_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \cdots & a_{nm} + b_{nm} \end{bmatrix} \end{aligned}$$

  - ie. coefficient $c_{ij} = a_{ij} + b_{ij}$

- multiplication by scalar
  - $$\mathbf{B} = c \times \mathbf{A} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1m} \\ ca_{21} & ca_{22} & \cdots & ca_{2m} \\ \vdots & \vdots & & \vdots \\ ca_{n1} & ca_{n2} & \cdots & ca_{nm} \end{bmatrix}$$
  - ie. coefficient $b_{ij} = ca_{ij}$
- identity matrix
  - $$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$
  - ie. ones on diagonal, zero elsewhere
  - only appropriate for *square matrices*
- zero matrix
  - $$\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$
  - ie. zeros everywhere
  - only appropriate for *square matrices*

- determinant
  - denoted as $|\ \mathbf{A}\ |$ or $det(\mathbf{A})$
  - eg. $\begin{vmatrix} a_{11} a_{12} \\ a_{21} a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$
  - equal to volume of box (parallelopiped) defined by row vectors

- transpose
  - if $\mathbf{A} = \begin{bmatrix} a_{ij} \end{bmatrix}$ then $\mathbf{A}^T = \begin{bmatrix} a_{ji} \end{bmatrix}$
  - ie. flip about diagonal
  - for vectors $\mathbf{a} \bullet \mathbf{b} = \mathbf{a}^T\mathbf{b}$

- dot product of vectors (scalar product)
  - $c = \mathbf{a} \bullet \mathbf{b} = \Sigma_{i=1}^{n} a_i b_i$
  - eg. if $\mathbf{a}^{\mathbf{T}} = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$ and $\mathbf{b}^{\mathbf{T}} = \begin{bmatrix} 3 & 1 & 6 \end{bmatrix}$ then

    $$c = \begin{bmatrix} 1\ 2\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix} = 1 \times 3 + 2 \times 1 + 2 \times 6 = 17$$

- multiplication of matrices
  - if $\mathbf{A} = \left[a_{ik}\right]$ $(m \times n)$ and $\mathbf{B} = \left[b_{kj}\right]$ $(n \times s)$ then

$$
\begin{aligned}
\mathbf{C} &= \mathbf{AB} \\
&= \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{ij} & \cdots & a_{1j} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & a_{mn} \end{bmatrix} \\
&= [c_{ij}]
\end{aligned}
$$

  - element c_{ij} is found by taking the dot product of the ith row vector of **A** and the jth column vector of **B**
  - ie. $c_{ij} = \left[a_{i1} \;\cdots\; a_{in}\right] \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix} = \Sigma_{k=1}^{n} a_{ik} b_{kj}$

- row operations

  A number of row operation may be performed on a matrix without altering the solution

  - row combinations (eg. subtract $n\times$ row i from row j)
  - swapping rows (eg. exchange row i with row j)
  - scalar multiplication (eg. multiply row i by c)

- inverse

  - if $\mathbf{Y} = \mathbf{AX}$ and we want to solve for **X**, then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{Y}$
  - $\mathbf{A}^{-1}$ is termed the inverse matrix of **A**
  - $\mathbf{A}^{-1}$ can be constructed whenever **A** is *non-singular*
  - the following conditions for *invertability* or *non-singularity* are equivalent
    * no one equation in the system can be expressed as a linear combination of the others
    * the determinant of **A** is non-zero

      ie. $\mid \mathbf{A} \mid \neq 0$
    * the columns (rows) of the coefficient matrix are *linearly independent*

Properties of Matrices

- $\mathbf{I}\,\mathbf{A} = \mathbf{A} = \mathbf{A}\mathbf{I}$
- $\mathbf{0} + \mathbf{A} = \mathbf{A} = \mathbf{A} + \mathbf{0}$
- $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
- $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$
- $\mathbf{A}\,(\mathbf{B}\mathbf{C}) = (\mathbf{A}\mathbf{B})\mathbf{C}$
- But $\mathbf{A}\mathbf{B} \neq \mathbf{B}\mathbf{A}$