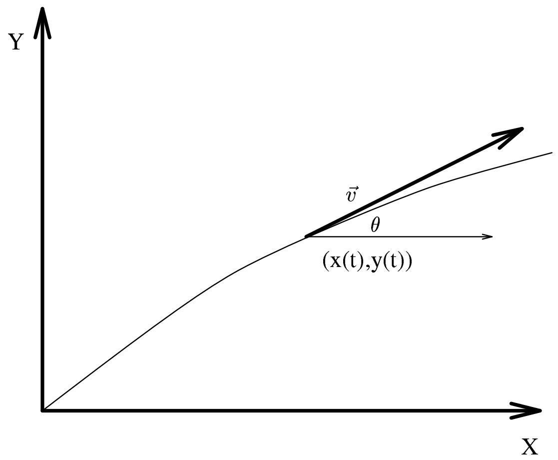
Mathematical Modeling

Modeling

- first problem is to formulate a suitable mathematical model of the problem
- may require cooperation of discipline specialists (eg. biochemists, physicists) along with mathematicians and computer scientists
- model based on general theories and laws of particular discipline
- specialize general laws to solve particular problem
- may need to simplify and approximate to ensure tractability (eg. in modeling a flight path of a GIS satellite, may ignore the gravitational force from the planets, which although present, is small)
- balance between what to include for completeness, and what to remove for tractability
- this is less of a problem as computers become more powerful, but the reduction of a model will always be an issue even if just from an economic and efficiency point of view

Trajectory of a rocket.



$$x(0) = y(0) = 0$$

$$\frac{dv}{dt} = \frac{1}{m}\left(T - \frac{1}{2}c\rho s v^2\right) - g\sin\theta - \frac{1}{m}.\frac{dm}{dt}v$$

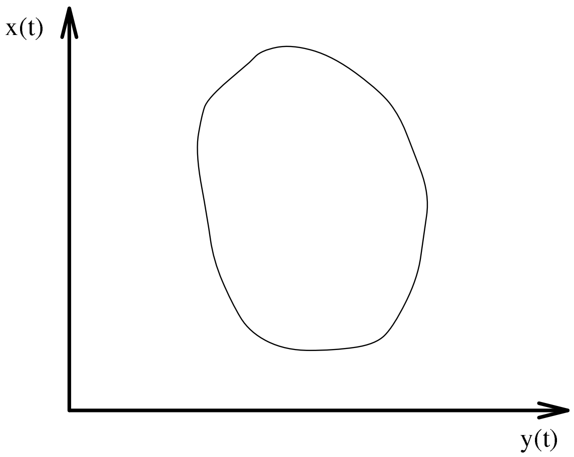
$$\frac{d\theta}{dt} = -\frac{g}{v}\cos\theta$$

$$\frac{dx}{dt} = v\cos\theta \qquad \frac{dy}{dt} = v\sin\theta \qquad \vec{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$$

where

m	=	mass
T	=	thrust
c	=	coefficient of drag
ρ	=	air density
s	=	cross-sectional area of rocket
g	=	acceleration due to gravity . . . etc. . . .

Predator-Prey.



$$\frac{dx}{dt} = \alpha x + \beta xy \quad \alpha > 0, \beta < 0$$

$$\frac{dy}{dt} = \gamma y + \delta xy \quad \gamma < 0, \delta > 0$$

where

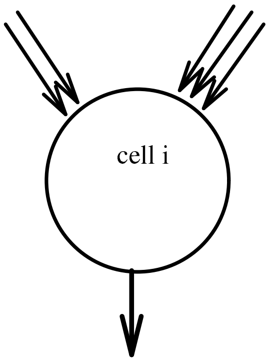
x	=	x(t)	=	number of prey (e.g. rabbit)
y	=	y(t)	=	number of predators (e.g. fox)

No known non-trivial analytical solution, so must use numerical methods. For example, discrete-time dynamical system:

$$x(t+1) = \alpha x(t) - \beta x(t)y(t)$$

$$y(t+1) = \gamma y(t) - \delta x(t)y(t)$$

Response of neuron.



$$\alpha \frac{dx_i}{dt}(t) = -x_i(t) + S \left\{ f_i(t) + \sum_{j=1}^{n} c_{ij} x_j(t - \sigma_{ij}) \right.$$

$$\left. + \sum_{j=1}^{n} b_{ik} \int_0^t x_i(\tau) e^{-a_{ik}(t-\tau)} d\tau \right\}$$

where

S(u)	=	$1/(1 + e^{-u})$
c_{ij}	=	excitatory/inhibitory lateral connections
σ_{ij}	=	associated time lags
b_{ik}	=	adaption coefficients
a_{ik}	=	temporal filter rate . . . etc. . . .

- as well as formulation of model, need to also consider initial or boundary conditions eg. latitude & longitude of satellite launch, fuel burn time, ...

- once a model has been developed, try to solve it

  - **analytic solution** - 'closed form' ie. solve the given model exactly using mathematical means eg. solve differential equation for the increase in population of a species

    $$\frac{dp(t)}{dt} = \alpha p(t), \ \ \alpha = constant$$

    If the population is p_0 at time t_0, then the solution is

    $$p(t) = p_0 e^{\alpha(t-t_0)}$$

    Using appropriate values, this gives good agreement with current world population growth. Analytic solutions are usually found only for simple models.

  - **numerical solution** - 'open form' ie. find an approximate solution using numerical methods