Integration/Quadrature
Aims Of This Section

- briefly look at *symbolic* integration
- examine standard *numerical* integration techniques
  - Classical Formulas
  - Gauss Integration
  - Cubic Interpolation
- discuss open and closed formulae
- examine accuracy of results
- briefly look at adaptive methods

Symbolic Integration

- A number of packages are available which perform *symbolic* integration, that is, they perform the *algebraic* manipulations to put a given function into an integrable form, then perform the integration analytically.

- These packages include **Lex, Maple** and **Mathematica**.

- For example, given the indefinite integral

  ```
  > int(x^3*cos(x), x);
  ```

  Maple will respond with

  ```
    3              2
   x  sin(x) + 3 x  cos(x) - 6 cos(x) - 6 x sin(x)
  ```

  That is,

  $$\int x^3 cos(x)\, dx \;=\; x^3 + 3x^2 cos(x) - 6cos(x) - 6xsin(x)$$

- Maple is able to obtain this result by using integration by parts (3 times), exploring various paths until simple (known) integrals are obtained, then backtracking to collect up the total result.

  (first $u = x^3, \;\; v = sin(x)$ then $u = x^2, \;\; v = cos(x)$ then $u = x, \;\; v = sin(x)$)

- standard (known) results are stored in a database, and new results may be added.

- Definite integrals and numeric integration may also be performed by Maple.

Numerical Integration

- In this section we consider numerical approximations of the definite integral

$$I = \int_a^b f(x)\ dx$$

  where f is a real-valued function

- The value of the integral may be interpreted as the area bounded by the curve $y = f(x)$ where $a \leq x \leq b$

- In many cases, the integral may be evaluated *analytically*, for example:

$$\int_a^b x^2\ dx \;=\; \left[\frac{x^3}{3}\right]_a^b \;=\; \frac{1}{3}(b^3 - a^3)$$

  Standard techniques may be used, such as

  substitutions ($x = sin(u)$)

  integration by parts ($\int u\ dv \;=\; uv \;-\; \int v\ du$)

- But in many cases analytic integration is not possible, either because the function is only known at a discrete set of points (*intensional form*), or because its known (*extensional*) form is too complex.

- In either case, *numerical integration* is required.

- The two cases are slightly different, however.
  - if the extensional (formula) form is known, appropriate function points may be calculated for any value of x
  - however, when the value of the function is known only at certain predetermined points, then these values must be used, and a form of interpolation is required to approximate the function at any intermediate x values

- One method used to obtain the integral is to approximate f by a function that is able to be analytically integrated (say a polynomial), and then perform the integration analytically to find the appropriate value.

- This method forms the basis of a number of standard methods. Done with care, this method can produce good results, but if done without thought may lead to serious errors.

- If we approximate $f(x)$ on the interval $[a, b]$ with a function $g(x)$ where

$$g(x) = \sum_{i=1}^{n} a_i \phi_i(x)$$

then

$$I(f) \simeq I(g) = \sum_{i=1}^{n} a_i \int_a^b \phi_i(x)\, dx$$

- For example, if $g(x)$ is a cubic spline approximation to f then $I(g)$ usually approximates $I(f)$ very well.

- Another method is to divide the interval $[a, b]$ into a number of columns (strips) whose areas can be approximated and summed.

- Most methods use a combination of these techniques.

- Any formula that approximates $I(f)$ is called a **numeric integration** or **quadrature** rule.

Classical Formulas

- First discuss several increasingly accurate methods. These early methods approximate the integral by dividing the interval $[a, b]$ into n strips each of width h

$$b = a + nh$$

- The larger n is (ie. the smaller h is), the more accurate the result – up to a point.

- The accuracy and efficiency of the various methods can be compared by determining how the error term (which includes both ***truncation*** and ***representation*** error) depends on h.

Rectangular Rule:

- The simplest method to evaluate the area under a curve is to treat it as a sum of adjacent narrow rectangles.

- Since there is no reason to give one side of the rectangle more importance than the other, define

$$\int_t^{t+h} f(x)dx \approx hf(t\ +\ h/2)$$

- In fact, by taking the height of each rectangle at the mid-point of each interval, the errors tend to cancel.

- The integral may then be estimated by the formula

$$R(h) = \int_a^b f(x)dx\ =\ h[f(a + h/2)\ +\ f(a + 3h/2)\ + \ldots$$

$$\ldots\ +\ f(b - h/2)]$$

- Note: programming tips
  - when programming, better to use $i * h$ in calculations than to add h repeatedly within a loop. This reduces the chance of representation errors.
  - if shape of function can be drawn, can then decide the order in which terms should be added. Sum smaller areas first to achieve greater accuracy.

Trapezoidal Rule:

- An *apparently* more accurate rule estimates the area of each strip by the area of a trapezium, in which the function $f(x)$ is replaced by a straight line passing through $f(a)$ and $f(b)$.

- In this case,

$$\int_a^b f(x)\, dx \;\simeq\; \frac{(b-a)}{2}(f(a)+f(b))$$

- Applying this rule for a number of strips gives the *Extended Trapezoidal Rule*. Noting

$$\int_t^{t+h} f(x)dx \approx \frac{h}{2}\left[f(t)\ +\ f(t+h)\right]$$

the integral may then be estimated by:

$$T(h) = \int_a^b f(x)dx = \frac{h}{2}\left[f(a)\ +\ 2f(a+h)\ +\ \ldots\right.$$

$$\left.\ldots\ +\ 2f(b-h)\ +\ f(b)\right]$$

Simpson's Rule:

- A slightly more complex rule with a much better accuracy can be derived by approximating adjacent parts of the curve f by quadratic functions rather than linear ones.

- Let the curve through x, $x+h$ and $x+2h$ satisfy the equation

$$f(x) = ax^2 \; + \; bx \; + \; c$$

for unknown coefficients a, b, c.

$$\int_t^{t+2h} f(x)dx \; = \; \left[ax^3/3 \; + \; bx^2/2 \; + \; cx\right]_t^{t+2h}$$

$$= \frac{h}{3}[f(t) \; + \; 4f(t+h) \; + \; f(t+2h)]$$

- The entire integral over multiple strips is then:

$$S(h) = \int_a^b f(x)dx$$

$$= \frac{h}{3}[f(a) \; + \; 4f(a+h) \; + \; 2f(a+2h) \; + \; \ldots$$

$$\ldots + \; 4f(b-h) \; + \; f(b)]$$