Computer Vision
cmput 428/615

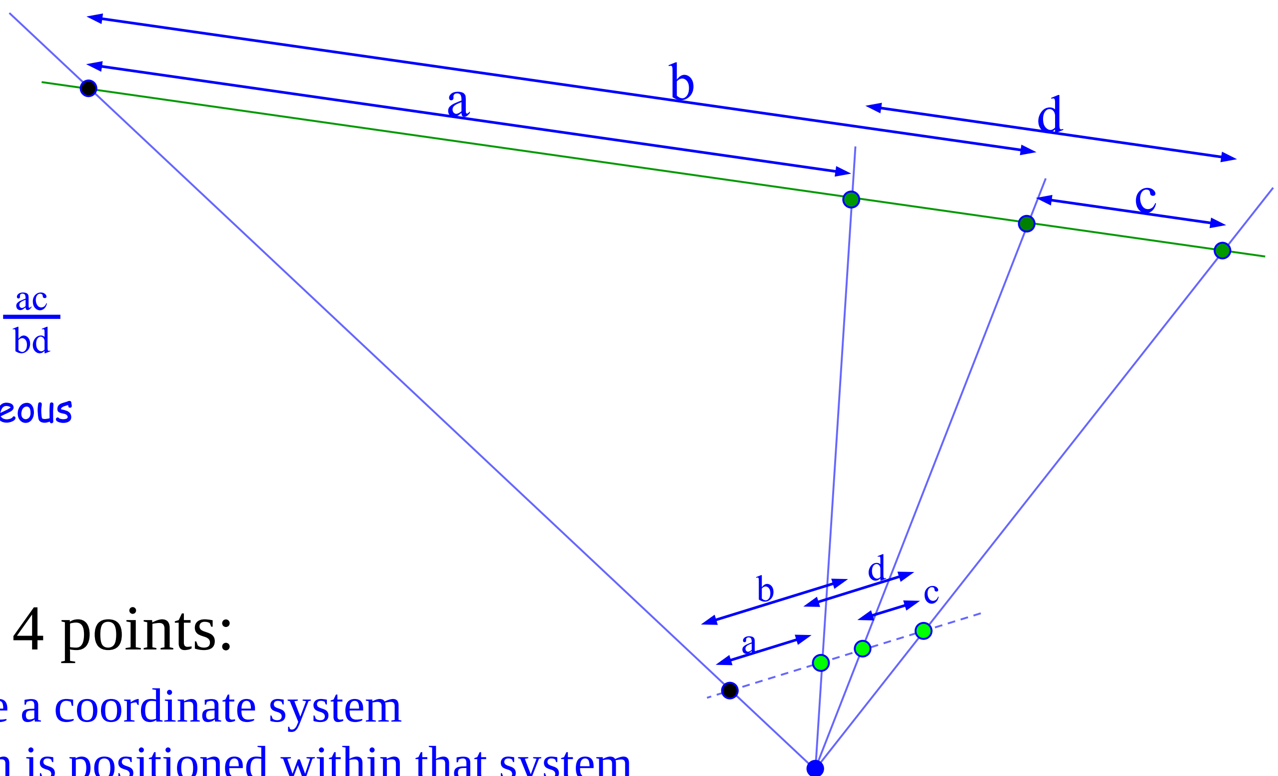
Lecture 8:

3D projective geometry and it’s applications

Martin Jagersand

First 1D Projective line
Projective Coordinates

- Basic projective invariant in P^1: the *cross ratio*



$$\frac{ac}{bd} = \frac{ac}{bd}$$

inhomogeneous

Requires 4 points:

3 to create a coordinate system

The fourth is positioned within that system

Cross ratio

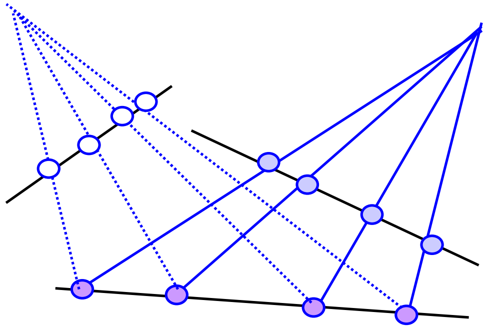
HZ CH 2.5

- Basic projective invariant in P^1: the *cross ratio*

$$\{\mathbf{x}_1, \mathbf{x}_2; \mathbf{x}_3, \mathbf{x}_4\} = \frac{|\mathbf{x}_1\mathbf{x}_2||\mathbf{x}_3\mathbf{x}_4|}{|\mathbf{x}_1\mathbf{x}_3||\mathbf{x}_2\mathbf{x}_4|} \qquad |\mathbf{x}_i\mathbf{x}_j| = \det\begin{bmatrix} x_{i1} & x_{j1} \\ x_{i2} & x_{j2} \end{bmatrix}$$

homogeneous

- Properties:
  - Defines coordinates along a 1d projective line
  - Independent of the homogeneous representation of **x**
  - Valid for ideal points
  - Invariant under homographies



Projective 3D space

- Points $\mathbf{X} = (x_1, x_2, x_3, x_4), \quad x_4 \neq 0$

  $(x_1, x_2, x_3, 0)$

  $(x_1, x_2, x_3, x_4) \rightarrow (x_1 / x_4, x_2 / x_4, x_3 / x_4)$

  $(X, Y, Z, 1) \leftarrow (X, Y, Z)$

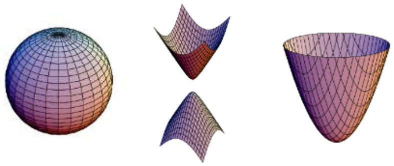
- Planes $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3, \pi_4)$

  $\boldsymbol{\pi}^T \mathbf{X} = 0$

Points and planes are dual in 3d projective space.

- Lines: 5DOF, various parameterizations
- Projective transformation:
  - 4x4 nonsingular matrix H
  - Point transformation $\mathbf{X}' = H\mathbf{X}$
  - Plane transformation $\boldsymbol{\pi}' = H^{-T} \boldsymbol{\pi}$
- Quadrics: Q $\mathbf{X}^T Q \mathbf{X} = 0$
  - 4x4 symmetric matrix Q
  - 9 DOF (defined by 9 points in general pose)

Dual: Q* $\boldsymbol{\pi}^T Q^* \boldsymbol{\pi} = 0$



3D points

3D point

$(X,Y,Z)^\mathsf{T}$ in $\mathbf{R}^3$

$\mathrm{X}=(X_1,X_2,X_3,X_4)^\mathsf{T}$ in $\mathbf{P}^3$

$$\mathrm{X}=\left(\frac{X_1}{X_4},\frac{X_2}{X_4},\frac{X_3}{X_4},1\right)^\mathsf{T}=(X,Y,Z,1)^\mathsf{T} \qquad (X_4\neq 0)$$

projective transformation

$$\mathrm{X}'=\mathbf{H}\,\mathrm{X}$$ (4x4-1=15 dof)

Planes

3D plane

$$\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 = 0$$

$$\pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3 + \pi_4 X_4 = 0$$

$$\pi^{\mathsf{T}} \mathrm{X} = 0$$

Transformation

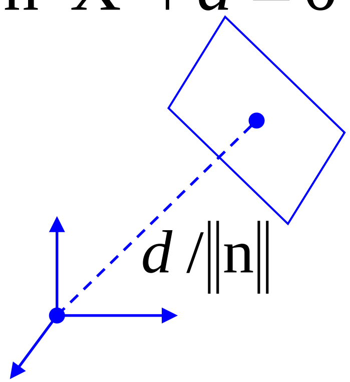
$$\mathrm{X}' = \mathbf{H}\,\mathrm{X}$$

$$\pi' = \mathbf{H}^{-\mathsf{T}}\,\pi$$

Euclidean representation

$$\mathrm{n}^T \tilde{\mathrm{X}} + d = 0 \qquad \mathrm{n} = (\pi_1, \pi_2, \pi_3)^{\mathsf{T}} \qquad \tilde{\mathrm{X}} = (X, Y, Z)^{\mathsf{T}}$$

$$\pi_4 = d \qquad X_4 = 1$$



Dual: points ↔ planes, lines ↔ lines

Planes from points

Solve π from $X_1^T\pi = 0$, $X_2^T\pi = 0$ and $X_3^T\pi = 0$

$$\begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \end{bmatrix}\pi = 0$$ (solve π as right nullspace of $\begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \end{bmatrix}$)

Or implicitly from coplanarity condition

$$\det\begin{bmatrix} X_1 & (X_1)_1 & (X_2)_1 & (X_3)_1 \\ X_2 & (X_1)_2 & (X_2)_2 & (X_3)_2 \\ X_3 & (X_1)_3 & (X_2)_3 & (X_3)_3 \\ X_4 & (X_1)_4 & (X_2)_4 & (X_3)_4 \end{bmatrix} = 0$$

$$\det[X\ X_1\ X_2\ X_3] = 0$$

$$X_1 D_{234} - X_2 D_{134} + X_3 D_{124} - X_4 D_{123} = 0$$

$$\pi = (D_{234}, -D_{134}, D_{124}, -D_{123})^T$$

Representing a plane by by lin comb of 3 points

Representing a plane by its nullspace span M

All points lin comb of basis

$$\mathrm{X} = \mathbf{M}\,\mathrm{x} \qquad \mathbf{M} = \left[\mathrm{X}_1 \mathrm{X}_2 \mathrm{X}_3\right]$$

$$\pi^{\mathsf{T}} \mathbf{M} = 0$$

Canonical form:

Given a plane $\pi = (a, b, c, d)^{\mathsf{T}}$

nullspace span M is $\mathbf{M} = \begin{bmatrix} \mathrm{p} \\ \mathrm{I} \end{bmatrix}$ $\qquad p = \left(-\frac{b}{a}, -\frac{c}{a}, -\frac{d}{a} \right)^{\mathsf{T}}$

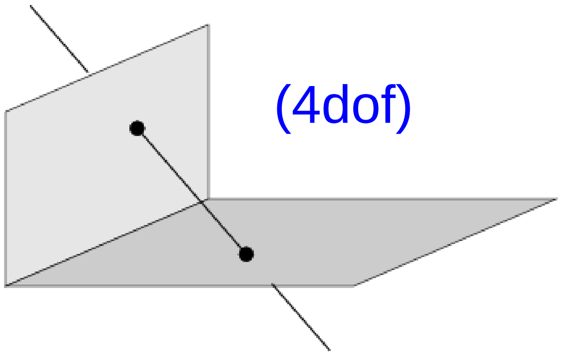
Points from planes

Solve X from $\pi_1^{\mathsf{T}}\mathrm{X} = 0$, $\pi_2^{\mathsf{T}}\mathrm{X} = 0$ and $\pi_3^{\mathsf{T}}\mathrm{X} = 0$

$$\begin{bmatrix} \pi_1^{\mathsf{T}} \\ \pi_2^{\mathsf{T}} \\ \pi_3^{\mathsf{T}} \end{bmatrix} \mathrm{X} = 0$$

(solve X as right nullspace of $\begin{bmatrix} \pi_1^{\mathsf{T}} \\ \pi_2^{\mathsf{T}} \\ \pi_3^{\mathsf{T}} \end{bmatrix}$)

Lines



(4dof)

Join of two points: A, B

$$W = \begin{bmatrix} A^\top \\ B^\top \end{bmatrix} \qquad \lambda A + \mu B$$

Intersection of two planes: P, Q

$$W^* = \begin{bmatrix} P^\top \\ Q^\top \end{bmatrix} \qquad \lambda P + \mu Q$$

Pencil of planes

$$W^* W^\top = W {W^*}^\top = 0_{2\times 2}$$

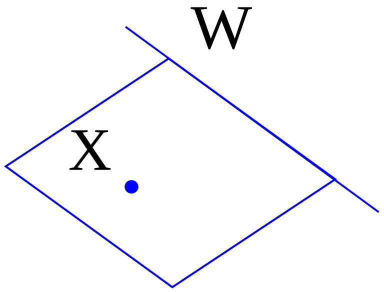
Example: X-axis

$$W = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad W^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Points, lines and planes

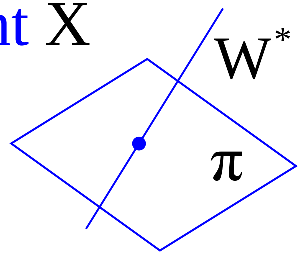
Join of point X and line W is plane π

$$\mathbf{M} = \begin{bmatrix} W \\ X^T \end{bmatrix} \qquad \mathbf{M}\pi = 0$$



Intersection of line W with plane π is point X

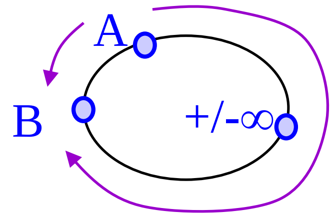
$$\mathbf{M} = \begin{bmatrix} W^* \\ \pi^T \end{bmatrix} \qquad \mathbf{M}X = 0$$

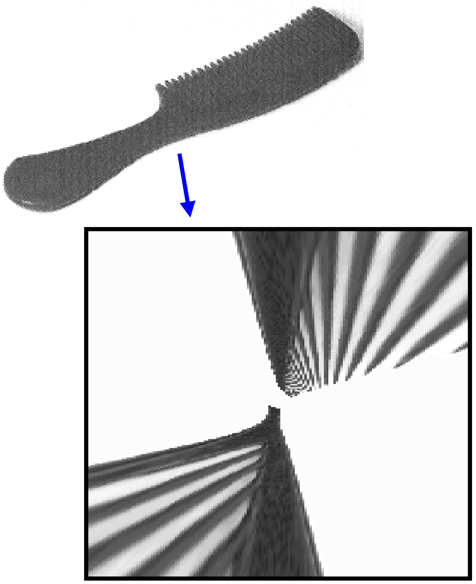


Affine space

Difficulties with a projective space:

- Nonintuitive notion of direction:
  - Parallelism is not represented
- Infinity not distinguished
- No notion of "inbetweenness":
  - Projective lines are topologically circular
- Only cross-ratios are available
  - Ratios are required for many practical tasks



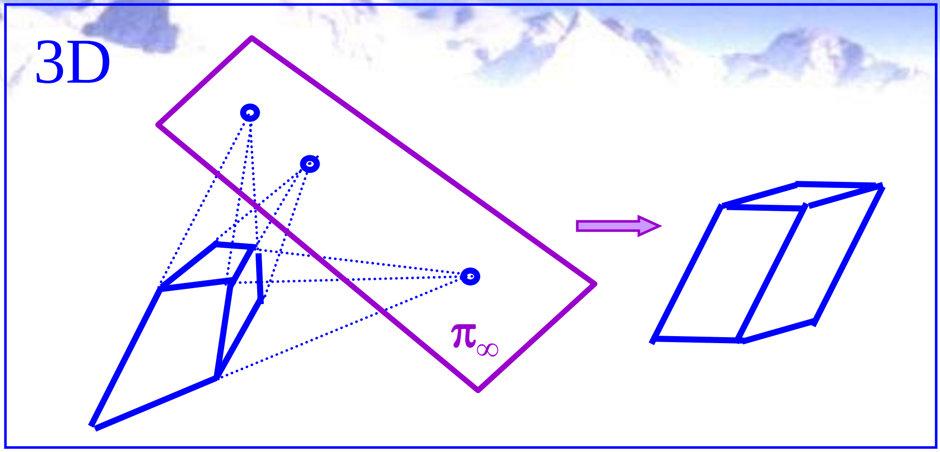


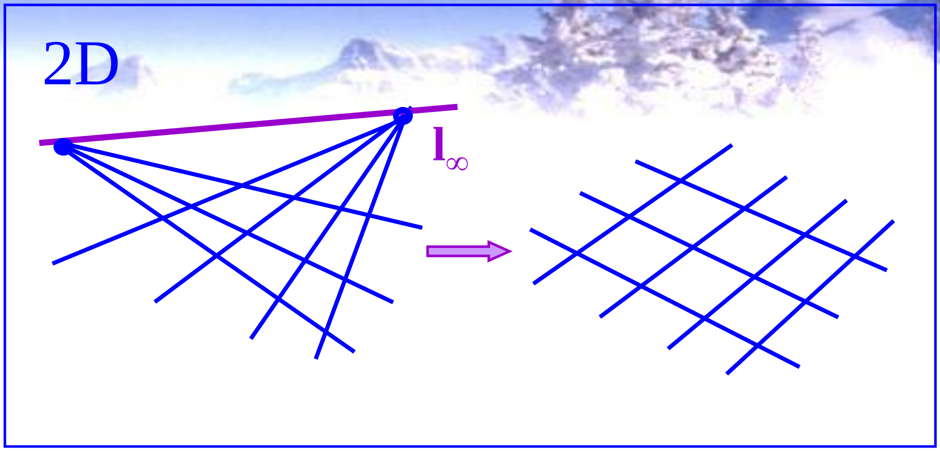
Solution: find the plane at infinity! $\boldsymbol{\pi}_\infty = (0,0,0,1)$

- Transform the model to give π_∞ its canonical coordinates
- 2D analogy: fix the horizon line $\mathbf{l}_\infty = (0,0,1)$

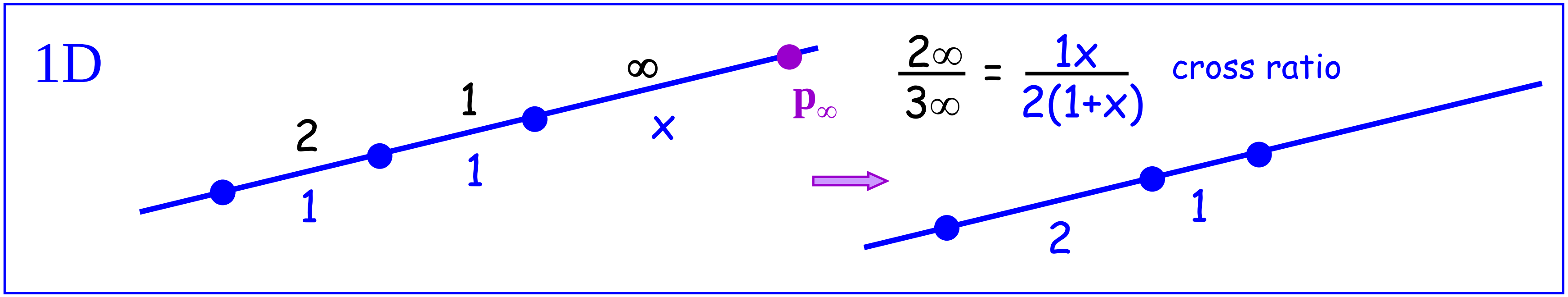
Determining the plane at infinity upgrades the geometry from projective to affine

From projective to affine





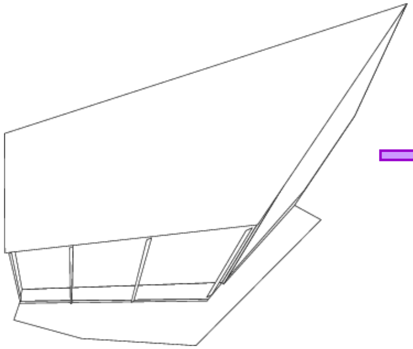
- Finding the plane (line, point) at infinity
  - 2 or 3 sets of parallel lines (meeting at "infinite" points)
  - a known ratio can also determine infinite points

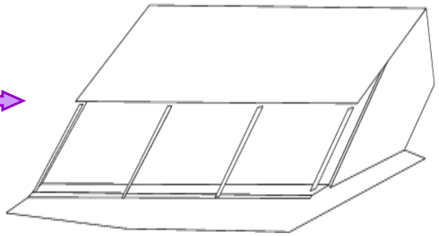


From projective to affine

Projective

Affine

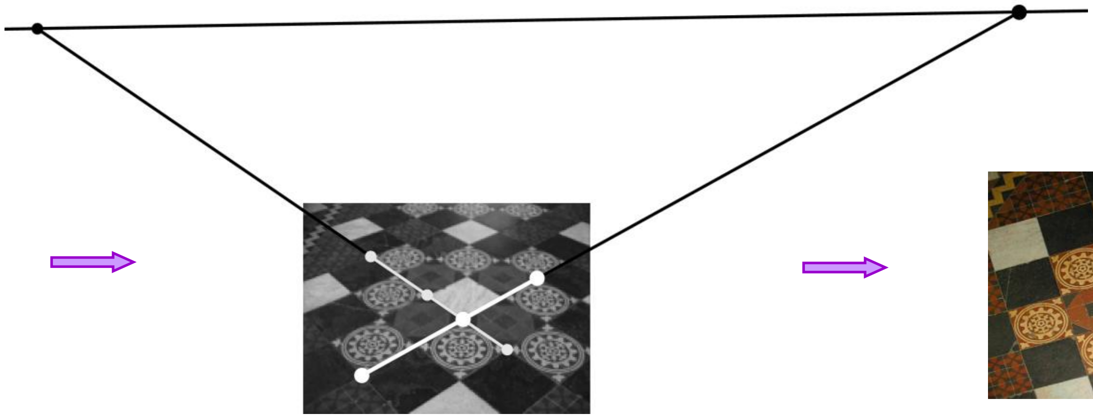




HZ

3D

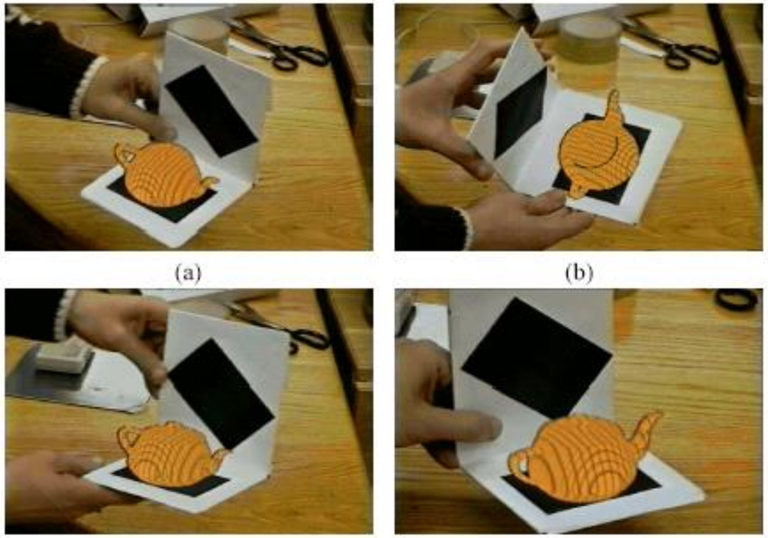
2D



Affine space

- Affine transformation
  - 12 DOF
  - Leaves π_∞ unchanged
- Invariants
  - Ratio of lengths on a line
  - Ratios of angles
  - Parallelism

$$H_A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Kutulakos and Vallino, *Calibration-Free Augmented Reality*, 1998

Metric space

- Metric transformation (similarity)
  - 7 DOF
  - Maps absolute conic to itself

$$H_S = \begin{bmatrix} sR & \mathbf{t} \\ \mathbf{O}^T & 1 \end{bmatrix}$$

- Invariants
  - Length ratios
  - Angles
  - The absolute conic

Without a yard stick, this is the highest level of geometric structure that can be retrieved from images

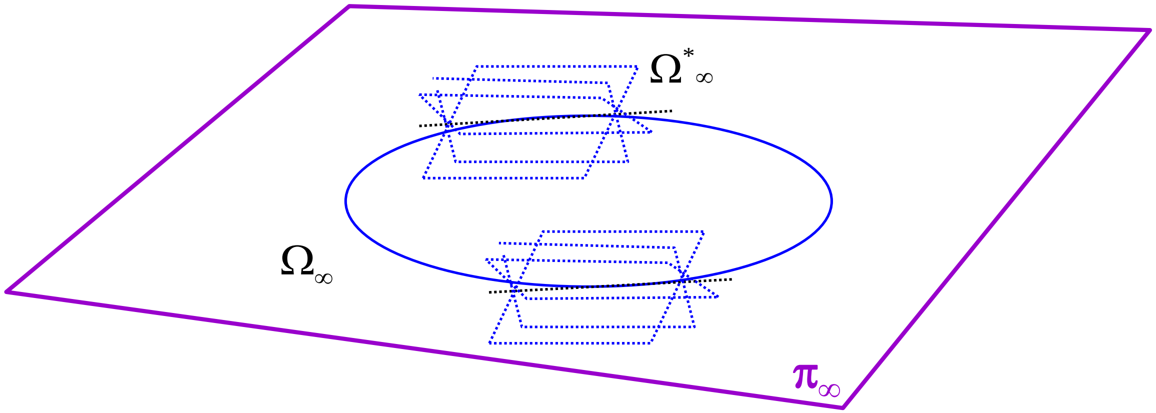
The absolute conic

- Absolute conic Ω_∞ is an imaginary circle on $\boldsymbol{\pi}_\infty$
- It is the intersection of every sphere with $\boldsymbol{\pi}_\infty$
- In a metric frame

$$\boldsymbol{\pi}_\infty = (0,0,0,1)$$

$$\Omega_\infty \quad \left.\begin{array}{r} x_1^2 + x_2^2 + x_3^2 \\ x_4 \end{array}\right\} = 0$$

$$\text{On } \boldsymbol{\pi}_\infty : \quad (x_1, x_2, x_3) I (x_1, x_2, x_3)^T = 0$$

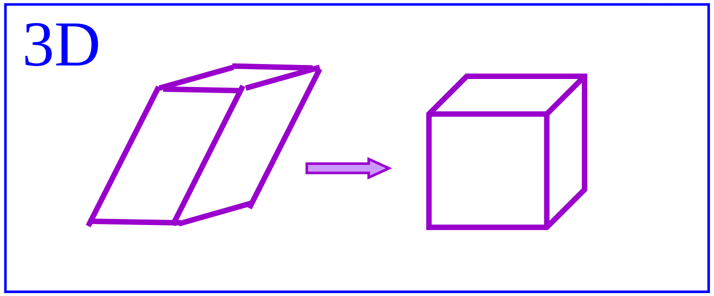


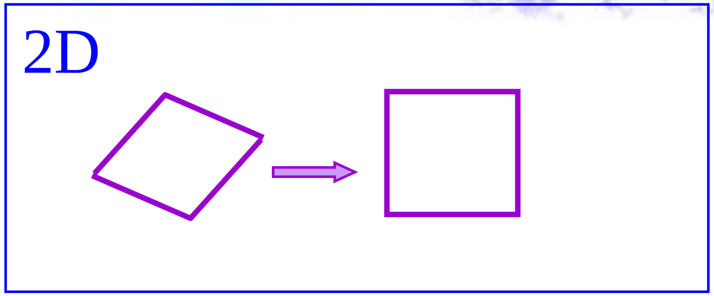
absolute dual quadric

$$\Omega^*_\infty = \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0}^T & 0 \end{bmatrix}$$

$$\pi^T \Omega^*_\infty \pi = 0$$

From affine to metric



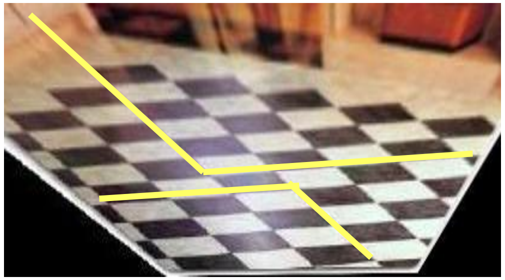


- Identify Ω_∞ on $\boldsymbol{\pi}_\infty$ OR identify Ω^*_∞
  - via angles, ratios of lengths
  - *e.g.* perpendicular lines $\mathbf{d}_1^T \Omega_\infty \mathbf{d}_2 = 0$
- Upgrade the geometry by bringing Ω_∞ to its canonical form via an affine transformation

1D ?

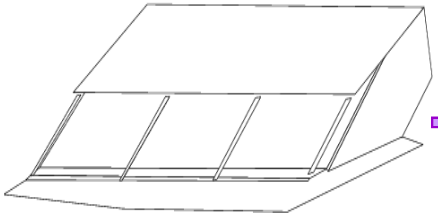
Examples

2D



Two pairs of perpendicular lines

3D



Affine

5 known points

Metric

What good is a projective model?

Represents fundamental feature interactions

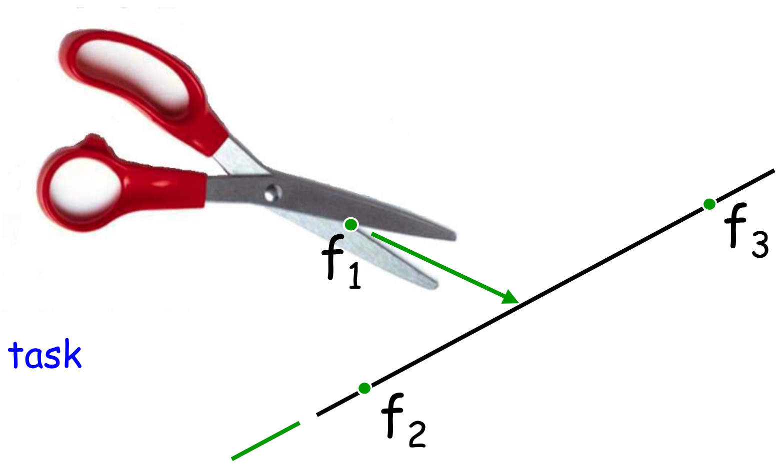
Used in rendering with an unconventional engine:

Physical Objects!

$$T_{col}(f_1,f_2,f_3) = \left\| f_1 - \overline{f_2 f_3} \right\|$$

Visual Servoing

Achieving 3d tasks
via
2d image control



collinearity task

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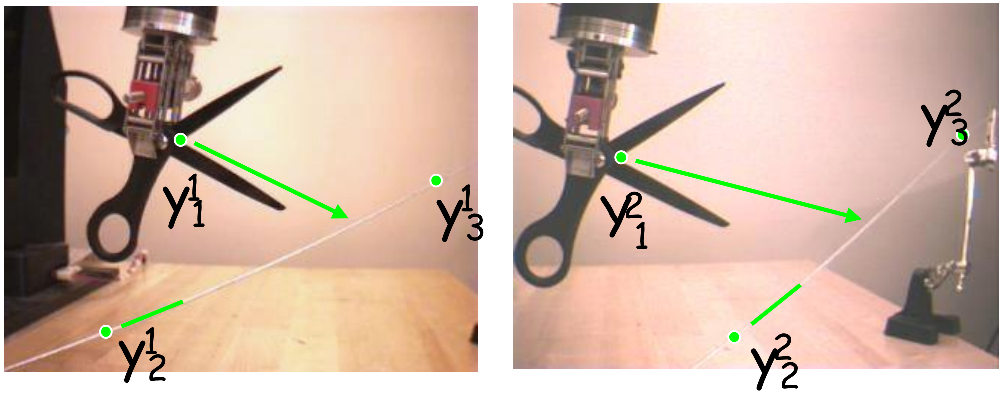
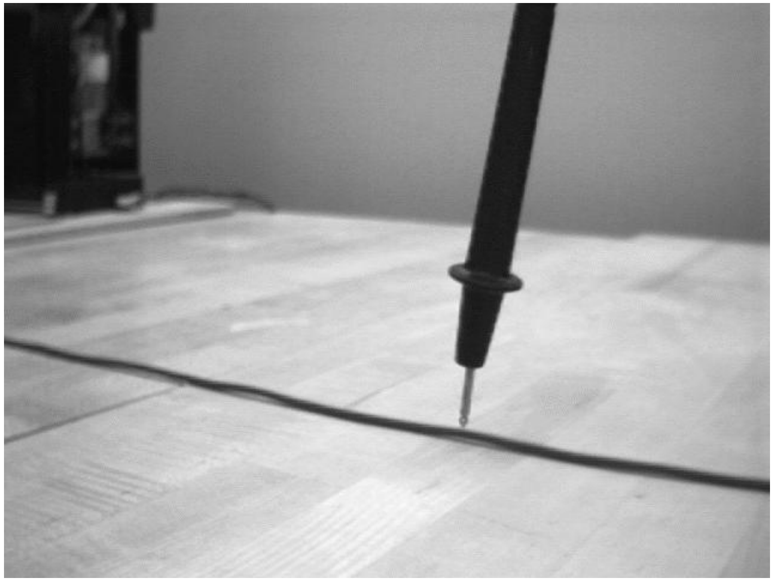


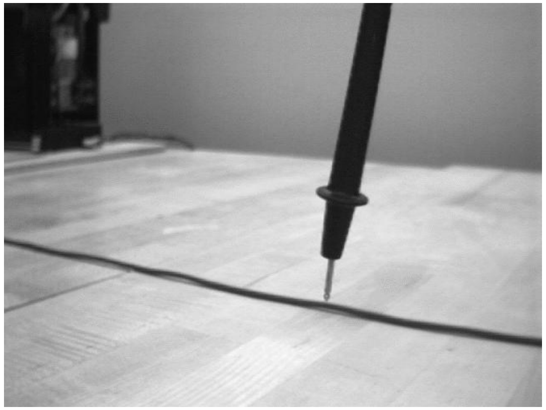
image collinearity constraint

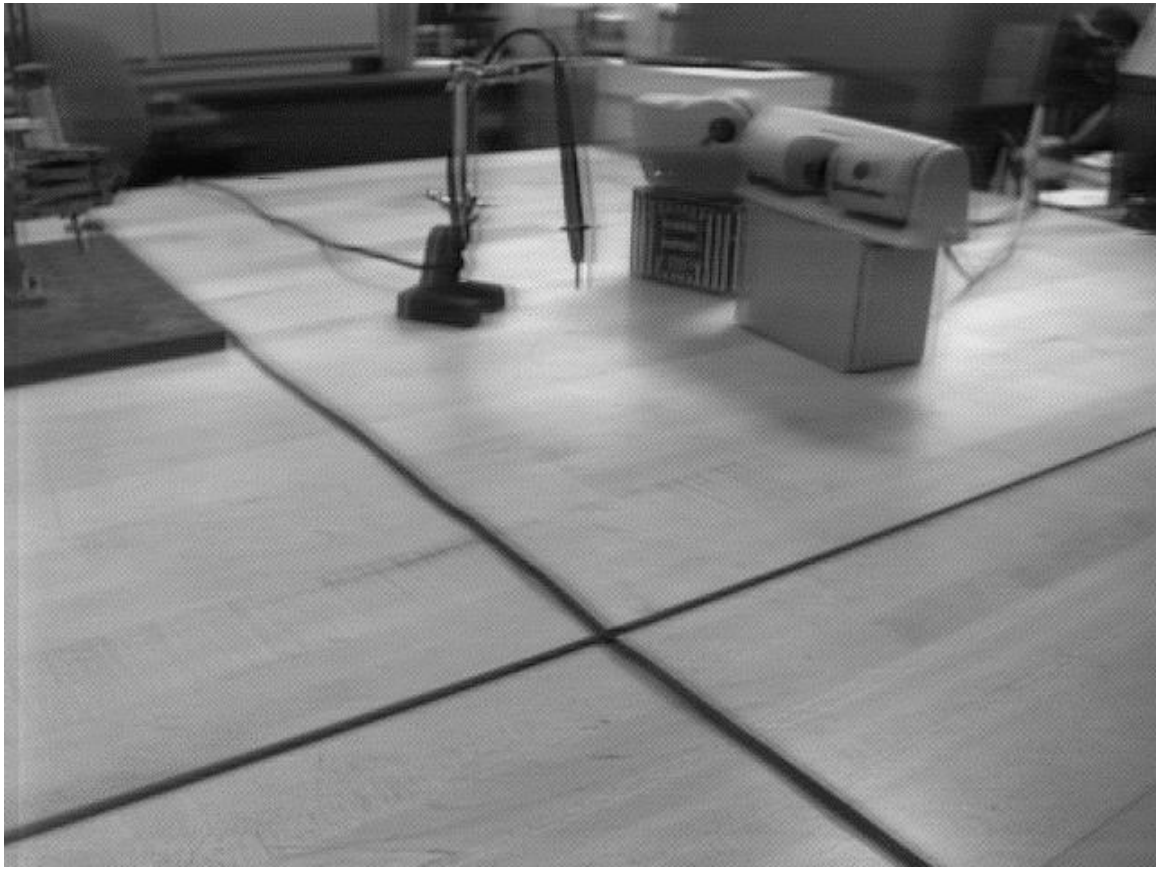
Collinearity?



A point meets a line in each of two images...

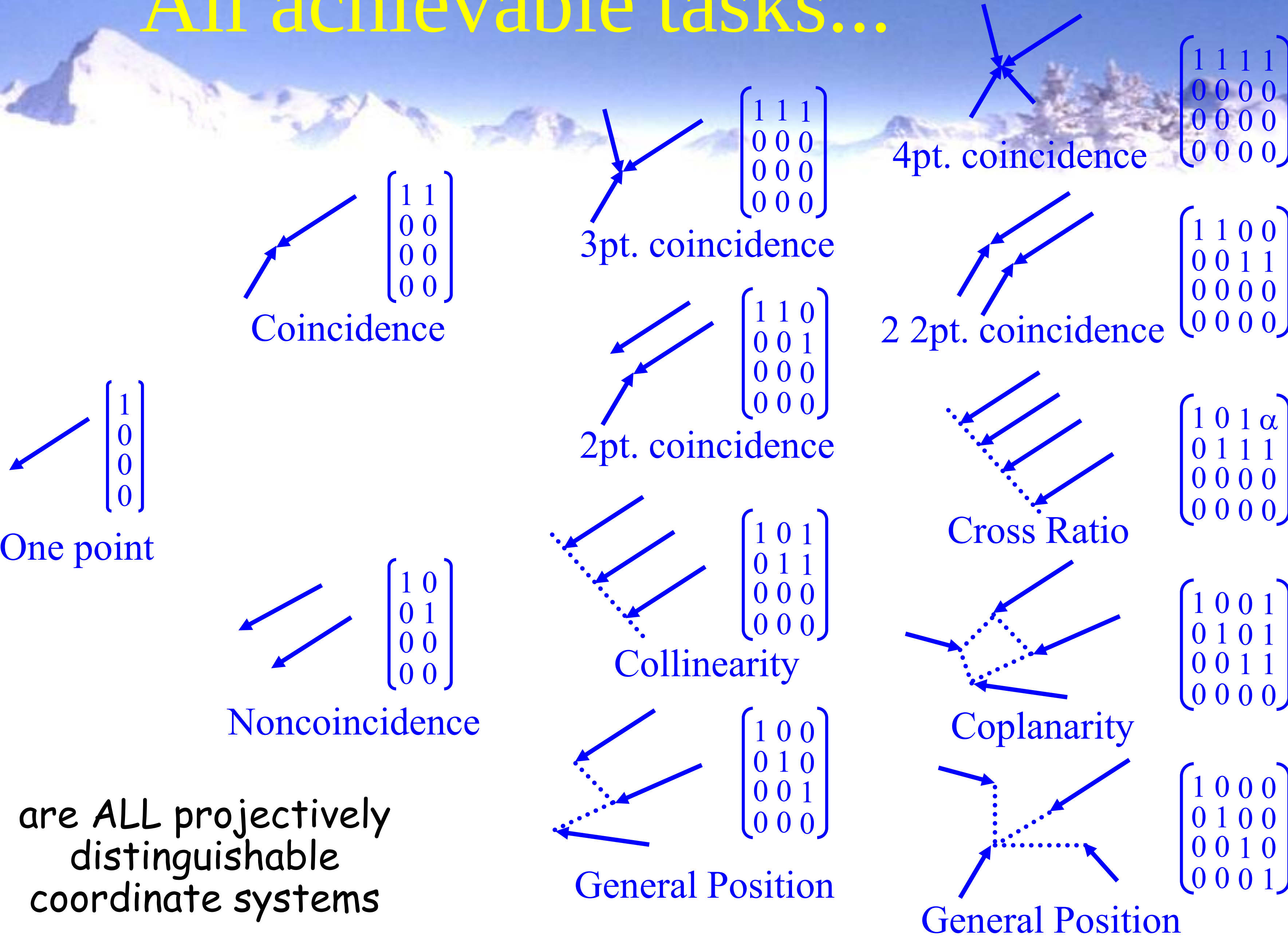
Collinearity?





But it doesn't guarantee task achievement !

All achievable tasks...



One point: $\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

Coincidence: $\begin{pmatrix}1&1\\0&0\\0&0\\0&0\end{pmatrix}$

Noncoincidence: $\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}$

3pt. coincidence: $\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}$

2pt. coincidence: $\begin{pmatrix}1&1&0\\0&0&1\\0&0&0\\0&0&0\end{pmatrix}$

Collinearity: $\begin{pmatrix}1&0&1\\0&1&1\\0&0&0\\0&0&0\end{pmatrix}$

General Position: $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{pmatrix}$

4pt. coincidence: $\begin{pmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$

2 2pt. coincidence: $\begin{pmatrix}1&1&0&0\\0&0&1&1\\0&0&0&0\\0&0&0&0\end{pmatrix}$

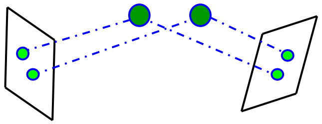
Cross Ratio: $\begin{pmatrix}1&0&1&\alpha\\0&1&1&1\\0&0&0&0\\0&0&0&0\end{pmatrix}$

Coplanarity: $\begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&1&1\\0&0&0&0\end{pmatrix}$

General Position: $\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$

are ALL projectively distinguishable coordinate systems

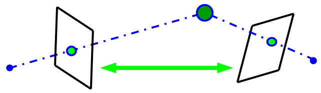
Composing possible tasks



Injective C_{inj}

T_{pp} point-coincidence

Task primitives



Projective C_{wk}

T_{pp}

T_{col} collinearity

T_{copl} coplanarity

$T_{cr\alpha}$ cross ratios (α)

AND

$$T_1 \wedge T_2 = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

OR

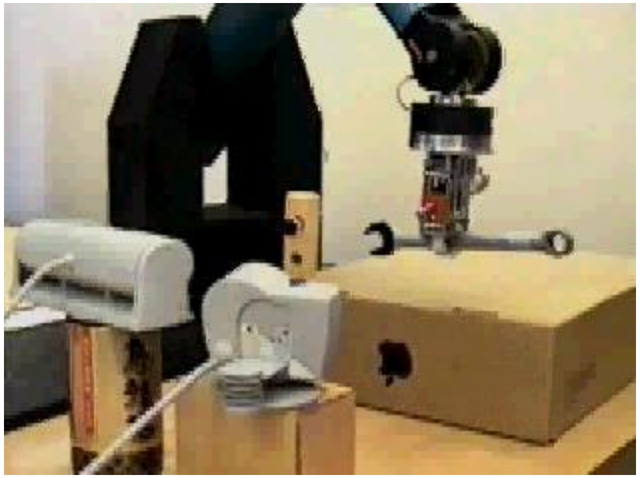
$$T_1 \vee T_2 = T_1 \cdot T_2$$

NOT

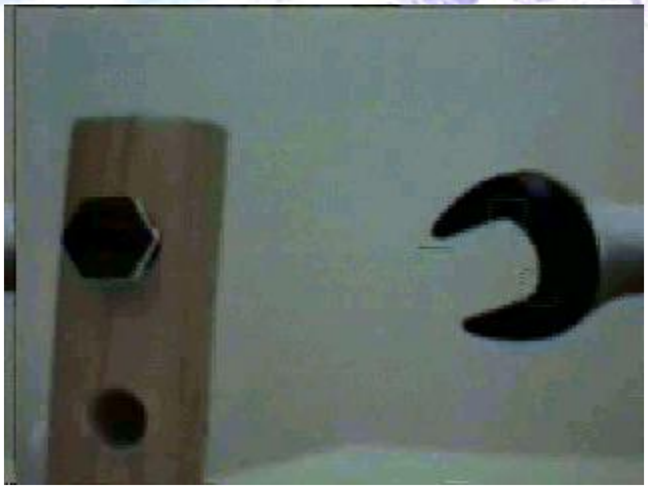
$$\neg T = \begin{cases} 0 & \text{if } T \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

Task operators

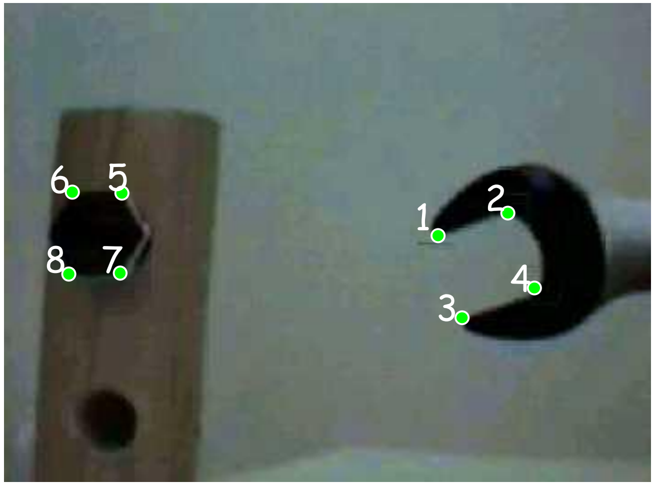
Result: Task toolkit



wrench: view 1

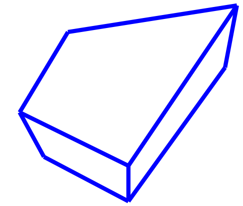
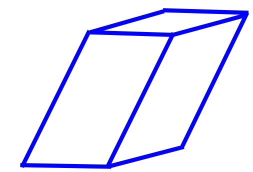
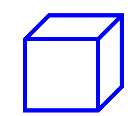
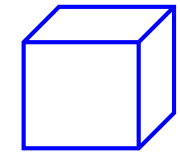


wrench: view 2



$$T_{wrench}(x_{1..8}) = \begin{array}{l} T_{pp}(x_1,x_5) \wedge \\ T_{pp}(x_3,x_7) \wedge \\ T_{col}(x_4,x_7,x_8) \wedge \\ T_{col}(x_2,x_5,x_6) \end{array}$$

Geometric strata: 3d overview

Group	Transformation	Invariants	Distortion
Projective 15 DOF	$H_P = \begin{bmatrix} A & \mathbf{t} \\ \mathbf{v}^T & v \end{bmatrix}$	• Cross ratio • Intersection • Tangency	
Affine 12 DOF	$H_A = \begin{bmatrix} A & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	• Parallelism • Relative dist in 1d • Plane at infinity $\boldsymbol{\pi}_\infty$	
Metric 7 DOF	$H_S = \begin{bmatrix} sR & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	• Relative distances • Angles • Absolute conic Ω_∞	
Euclidean 6 DOF	$H_E = \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$	• Lengths • Areas • Volumes	

3DOF $\boldsymbol{\pi}_\infty$

5DOF Ω_∞

Faugeras '95

Perspective and projection

- Euclidean geometry is not the only representation
  - for building models from images
  - for building images from models

- But when 3d realism is the goal,
  how can we effectively build and use

  projective
  affine
  metric
  } representations ?