CMPUT 340 (B1)

MIDTERM EXAMINATION

February 18, 2000

Instructor: **I. E. Leonard**

Instructions: The exam is closed book. Use of a calculator is permitted. You should attempt all five questions, since the total possible score is **55** points, and the exam will be graded out of **50** points. All problems have the weights shown.

Question 1. [5+5 points]

In the normalized floating point number system $F(\beta, t, L, U)$ we can determine exactly how many significant digits are lost in the subtraction $x - y$ when x is close to y (here the closeness of x to y is conveniently measured by $|1 - y/x|$. The theorem on loss of precision is as follows:

Theorem. If x and y are normalized floating point numbers in $F(\beta, t, L, U)$ such that $x > y > 0$, and if $\beta^{-p} \leq 1 - y/x \leq \beta^{-q}$ for some positive integers p and q, then at most p and at least q significant digits are lost in the subtraction $x - y$.

(a) To six decimal places, $\cos(\frac{1}{4}) = 0.968912$. Using the loss of precision theorem above, what can you say about the number of *binary* bits lost in the subtraction $1 - \cos(\frac{1}{4})$?

(b) If $f(x) = 1 - \cos x$ is to be evaluated at $x = 0$, show that this problem is well-conditioned.

Question 2. [5+5 points]

Working in the floating point number system $F(10, 3, -100, 100)$ with rounding, use Gaussian elimination to solve the system of linear equations

$$\begin{aligned} 0.2x_1 + 1.2x_2 + 1.6x_3 &= 4.5 \\ 1.2x_1 + 7.1x_2 - 6.0x_3 &= -4.3 \\ 4.2x_1 - 2.8x_2 + 10x_3 &= 5.3 \end{aligned}$$

(a) without pivoting,

(b) with partial pivoting.

Question 3. [10 points]

Find an $L\,U$ factorization of the matrix

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{pmatrix}$$

where U is *unit* upper triangular and L is lower triangular.

Question 4. [10 points]

Let $||\cdot|| : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a norm on the vector space $\mathbb{R}^n$, and Let $||\cdot|| : \mathbb{M}_{n\times n} \longrightarrow \mathbb{R}$ be the induced matrix norm on the ring of $n \times n$ real matrices. Prove that if $A \in \mathbb{M}_{n\times n}$ and there exists a scalar $\lambda > 0$ such that $||A\mathbf{x}|| \geq \lambda||\mathbf{x}||$ for all $\mathbf{x} \in \mathbb{R}^n$, then A is nonsingular and

$$||A^{-1}|| \leq \frac{1}{\lambda}.$$

Question 5. **[5+5+5 points]**

In this problem we consider the problem of evaluating the integrals

$$I_n = \int_0^1 \frac{x^n}{a+x} dx$$

for a fixed value of $a > 1$ and for $n = 0, 1, 2, \cdots$.

Note that since $x^{n+1} < x^n$ for $0 < x < 1$, then $0 < I_{n+1} < I_n$ for all $n \geq 0$, thus the sequence $\{I_n\}_{n \geq 0}$ is monotone decreasing and bounded below, hence converges. In fact, since

$$\frac{1}{(n+1)(a+1)} = \int_0^1 \frac{x^n}{1+a} dx < I_n < \int_0^1 \frac{x^n}{a} dx = \frac{1}{(n+1)a},$$

then $\lim_{n \to \infty} I_n = 0$.

Since

$$I_n = \int_0^1 \frac{x^{n-1}(x+a-a)}{x+a} dx = \frac{1}{n} - aI_{n-1}$$

for $n \geq 1$, then the sequence $\{I_n\}$ satisfies the discrete initial value problem

$$I_n = \frac{1}{n} - aI_{n-1}, \quad n \geq 1$$
$$I_0 = \log\left(\frac{1+a}{a}\right).$$

(a) Now suppose that I_0 is in error by the quantity ϵ_0, assuming all subsequent arithmetical operations are performed exactly, let $\widehat{I}_n$ be the values computed using the wrong starting value, then $\widehat{I}_0 = I_0 + \epsilon_0$, and

$$\widehat{I}_n = -a\widehat{I}_{n-1} + \frac{1}{n}.$$

Let $r_n = \widehat{I}_n - I_n$ be the error made at the n^{th} step. Find a discrete initial value problem satisfied by r_n.

(b) Solve the initial value problem for r_n and show this algorithm is unstable for $a >> 1$.

(c) Find a stable algorithm for computing the integrals I_n for $n \geq 0$, that is, one for which $\lim_{n \to \infty} r_n = 0$.