CMPUT 340 (B1)

FINAL EXAMINATION

April 18, 2000

Instructor: I. E. Leonard **Time: Two Hours**

Instructions: The exam is closed book. Use of a calculator is permitted. You should attempt all seven questions, since the total possible score is **80** points, and the exam will be graded out of **70** points. All problems have the weights shown.

Question 1. [10 points]

In 1685 John Wallis published a book called *Algebra*, in which he described a method devised by Newton for solving equations. In slightly modified form, this method was also published by Joseph Raphson in 1690. This form is the one now commonly called Newton's method or the Newton-Raphson method. Newton himself discussed the method in 1669 and illustrated it with the equation $x^3 - 2x - 5 = 0$. Wallis used the same example. Find a root of this equation at least to the precision of your calculator, thus continuing the tradition that every numerical analysis student should solve this venerable equation.

Question 2. [15 points]

If $a > 0$, then the iterative scheme for finding $\sqrt{a}$ using Newton's method is

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

for $n = 1, 2, \ldots$.

(a) Perform three iterations of this scheme for computing $\sqrt{2}$, starting with $x_0 = 1$.

(b) Perform three iterations of the bisection method for $\sqrt{2}$, starting with the interval $[1, 2]$.

(c) How many iterations are needed for each method in order to obtain an accuracy of 10^{-6}?

Question 3. [10 points]

Find a polynomial $p(x)$ of degree at most 3 such that

$$p(0) = 1, \quad p(1) = 0, \quad p'(0) = 0, \quad p'(-1) = -1.$$

Sketch the graph of the polynomial.

Question 4. [10 points]

Simple polynomial interpolation in two dimensions is not always possible. For example, suppose that the following data are to be represented by a polynomial of first degree in x and y

$$f(1,1) = 3, \quad f(3,2) = 2, \quad f(5,3) = 6.$$

Show that it is not possible to find a polynomial

$$p(x, y) = a + bx + cy$$

which interpolates f at these points.

Question 5. [10 points]

The Lagrange form of the polynomial that interpolates the function $f(x)$ at the distinct points $x_0, x_1, \ldots, x_n$ is given by

$$p(x) = \sum_{j=0}^{n} f(x_j)L_j(x),$$

where the *cardinal functions* $L_j(x)$ are polynomials of degree n with the property that $L_j(x_i) = 0$ for $i \neq j$ and $L_j(x_i) = 1$ for $i = j$.

(a) Find the cardinal functions $L_j(x)$ for $j = 0, 1, \ldots, n$.

(b) What is the sum of the cardinal functions, that is, what is $\sum_{j=0}^{n} L_j(x)$?

Question 6. [10 points]

The Newton form of the polynomial that interpolates the function $f(x)$ at the distinct points $x_0, x_1, \ldots, x_n$ is given by

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

the coefficient a_k is called the *divided difference of order k*, and is denoted by $f[x_0, x_1, \ldots, x_k]$.

(a) Let $q(x)$ be the polynomial of degree at most k which interpolates f at the points $x_0, x_1, \ldots, x_{k-1}$, and let $r(x)$ be the polynomial of degree at most k which interpolates f at $x_1, x_2, \ldots, x_k$. Show that the polynomial $p_k(x)$ given by

$$p_k(x) = q(x) + \frac{x - x_0}{x_k - x_0}[r(x) - q(x)]$$

has degree at most k and interpolates f at $x_0, x_1, \ldots, x_{k-1}, x_k$.

(b) Show that the divided differences satisfy the following recursive property:

$$f[x_0, x_1, \ldots, x_k] = \frac{f[x_1, x_2, \ldots, x_k] - f[x_0, x_1, \ldots, x_{k-1}]}{x_k - x_0}, \quad k \geq 1,$$

where $f[x_i] = f(x_i)$ for $i = 0, 1, \ldots, n$.

Question 7. [15 points]

Recall that a function g is said to be a *contraction mapping* on an interval $[a, b]$ if

(i) $g(x) \in [a, b]$ whenever $x \in [a, b]$, and

(ii) g satisfies a Lipschitz condition on $[a, b]$ with Lipschitz constant $0 < L < 1$, that is,

$$|g(x) - g(y)| \leq L|x - y|$$

whenever $x, y \in [a, b]$.

Show that if there is an interval $[a, b]$ on which g is a contraction mapping, then

(a) g has a unique fixed point $\alpha \in [a, b]$, that is, a unique α such that $g(\alpha) = \alpha$.

(b) For any $x_0 \in [a, b]$, the sequence defined by

$$x_{n+1} = g(x_n), \quad n = 0, 1, 2, \ldots,$$

converges to α.