

Computing Science 272 (B3)

Final Examination

Instructor: I.E. Leonard

Date: April 27, 2000

Time: Two Hours

Instructions:

1. This examination is open book and notes.
2. Communicating with anyone except the examination supervisor is prohibited.
3. Answer all questions on the examination.

Question 1. [10 points]

Let A be a subset of a universal set U , the *characteristic function* of A is the function χ_A from U to $\{0, 1\}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

- (a) Is χ_A ever a one-to-one function? Justify your answer.
- (b) Is χ_A ever an onto function? Justify your answer.

Question 2. [15 points]

Let $f : X \rightarrow Y$ be a function with domain X and codomain Y , and let $A \subseteq X$. The *image* of A under f is defined to be the subset of Y given by

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \in A\}.$$

Let A and B be subsets of the set X , show that

- (a) $f(A \cup B) = f(A) \cup f(B)$
- (b) $f(A \cap B) \subseteq f(A) \cap f(B)$.
- (c) Give an example to show that the inclusion in part (b) may be proper.

Question 3. [15 points]

Let \mathcal{R} be a relation on a set A .

- (a) Show that \mathcal{R} is symmetric if and only if $\mathcal{R} = \mathcal{R}^{-1}$, where \mathcal{R}^{-1} is the inverse relation.
- (b) Show that \mathcal{R} is antisymmetric if and only if $\mathcal{R} \cap \mathcal{R}^{-1}$ is a subset of the diagonal relation

$$\Delta = \{(a, a) \mid a \in A\}.$$

- (c) Show that if \mathcal{R} is reflexive and transitive, then $\mathcal{R}^n = \mathcal{R}$ for all positive integers n , where \mathcal{R}^n is the n -fold composition of \mathcal{R} with itself.

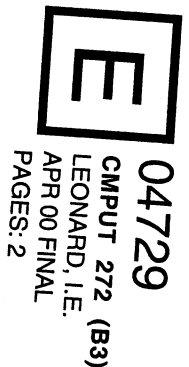
Question 4. [20 points]

Let \mathcal{R} be an equivalence relation on a set A , and, for $a \in A$, let $[a]$ be the equivalence class containing a , that is,

$$[a] = \{b \in A \mid (a, b) \in \mathcal{R}\}.$$

Show that the following are equivalent

- (a) $a\mathcal{R}b$
- (b) $[a] = [b]$
- (c) $[a] \cap [b] \neq \emptyset$.



Question 5. [10 points]

Given a set A , a relation $\mathcal{R} \subseteq A \times A$ is called a *partial ordering* on A if and only if it is reflexive, transitive, and antisymmetric.

Let $A = \mathbb{N}$ be the set of positive integers, and define a relation \mathcal{R} on \mathbb{N} as follows: If a and b are positive integers, then $(a, b) \in \mathcal{R}$ if and only if

- (i) a is odd and b is even, or
- (ii) a and b are both odd or both even, and $a \leq b$.

(a) Show that \mathcal{R} is partial ordering on \mathbb{N} .

(b) If $a < b$ means that $(a, b) \in \mathcal{R}$, list the elements of \mathbb{N} according to their ordering with respect to this partial order, for example, $n_1 < n_2 < n_3 < \dots$.

Question 6. [10 points]

Show that among any $n + 1$ positive integers a_1, a_2, \dots, a_{n+1} , not exceeding $2n$, there must be an integer that divides one of the other integers.

Hint: From the Fundamental Theorem of Arithmetic, any positive integer m can be written uniquely as a power of two times an odd integer, $m = 2^k \cdot q$, where k is a nonnegative integer and q is odd.

Write $a_j = 2^{k_j} \cdot q_j$, then each q_j is an odd integer less than $2n \dots$

Question 7. [10 points]

It is not known whether the sequence of integers

$$2^{2^n} + 3, \quad n = 1, 2, \dots$$

contains infinitely many primes.

Show, however, that *all* of the integers

$$2^{2^{2^k+1}} + 3, \quad k = 0, 1, 2, \dots$$

are divisible by 7.

Question 8. [10 points]

The Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$ satisfies the recurrence relation and the initial conditions

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1}, \quad n \geq 1, \\ F_0 &= 0, \\ F_1 &= 1. \end{aligned}$$

(a) By examining the differences $F_{n+1}F_{n-1} - F_n^2$, for small values of n , make a conjecture about the value of this expression.

(b) Prove your conjecture using the principle of mathematical induction.