Computing Science 272 (B3)

Final Examination

Instructor: I.E. Leonard

Date: April 27, 2000 **Time:** Two Hours

Instructions:

1. This examination is open book and notes.
2. Communicating with anyone except the examination supervisor is prohibited.
3. Answer all questions on the examination.

Question 1. [10 points]

Let A be a subset of a universal set U, the *characteristic function* of A is the function χ_A from U to $\{0, 1\}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

(a) Is χ_A ever a one-to-one function? Justify your answer.

(b) Is χ_A ever an onto function? Justify your answer.

Question 2. [15 points]

Let $f : X \longrightarrow Y$ be a function with domain X and codomain Y, and let $A \subseteq X$. The *image* of A under f is defined to be the subset of Y given by

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \in A\}.$$

Let A and B be subsets of the set X, show that

(a) $f(A \cup B) = f(A) \cup f(B)$

(b) $f(A \cap B) \subseteq f(A) \cap f(B)$.

(c) Give an example to show that the inclusion in part (b) may be proper.

Question 3. [15 points]

Let $\mathcal{R}$ be a relation on a set A.

(a) Show that $\mathcal{R}$ is symmetric if and only if $\mathcal{R} = \mathcal{R}^{-1}$, where $\mathcal{R}^{-1}$ is the inverse relation.

(b) Show that $\mathcal{R}$ is antisymmetric if and only if $\mathcal{R} \cap \mathcal{R}^{-1}$ is a subset of the diagonal relation

$$\Delta = \{(a, a) \mid a \in A\}.$$

(c) Show that if $\mathcal{R}$ is reflexive and transitive, then $\mathcal{R}^n = \mathcal{R}$ for all positive integers n, where $\mathcal{R}^n$ is the n-fold composition of $\mathcal{R}$ with itself.

Question 4. [20 points]

Let $\mathcal{R}$ be an equivalence relation on a set A, and, for $a \in A$, let $[a]$ be the equivalence class containing a, that is,

$$[a] = \{b \in A \mid (a, b) \in \mathcal{R}\}.$$

Show that the following are equivalent

(a) $a\mathcal{R}b$

(b) $[a] = [b]$

(c) $[a] \cap [b] \neq \emptyset$.

Question 5. [10 points]

Given a set A, a relation $\mathcal{R} \subseteq A \times A$ is called a *partial ordering* on A if and only if it is reflexive, transitive, and antisymmetric.

Let $A = \mathbb{N}$ be the set of positive integers, and define a relation $\mathcal{R}$ on $\mathbb{N}$ as follows: If a and b are positive integers, then $(a, b) \in \mathcal{R}$ if and only if

(i) a is odd and b is even, or

(ii) a and b are both odd or both even, and $a \leq b$.

(a) Show that $\mathcal{R}$ is partial ordering on $\mathbb{N}$.

(b) If $a \prec b$ means that $(a, b) \in \mathcal{R}$, list the elements of $\mathbb{N}$ according to their ordering with respect to this partial order, for example, $n_1 \prec n_2 \prec n_3 \prec \cdots$.

Question 6. [10 points]

Show that among any $n + 1$ positive integers $a_1, a_2, \ldots, a_{n+1}$, not exceeding $2n$, there must be an integer that divides one of the other integers.

Hint: From the Fundamental Theorem of Arithmetic, any positive integer m can be written uniquely as a power of two times an odd integer, $m = 2^k \cdot q$, where k is a nonnegative integer and q is odd.

Write $a_j = 2^{k_j} \cdot q_j$, then each q_j is an odd integer less than $2n$ ....

Question 7. [10 points]

It is not known whether the sequence of integers

$$2^{2^n} + 3, \qquad n = 1, 2, \cdots$$

contains infinitely many primes.

Show, however, that *all* of the integers

$$2^{2^{2k+1}} + 3, \qquad k = 0, 1, 2, \cdots$$

are divisible by 7.

Question 8. [10 points]

The Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... satisfies the recurrence relation and the initial conditions

$$F_{n+1} = F_n + F_{n-1}, \quad n \geq 1,$$
$$F_0 = 0,$$
$$F_1 = 1.$$

(a) By examining the differences $F_{n+1}F_{n-1} - F_n^2$, for small values of n, make a conjecture about the value of this expression.

(b) Prove your conjecture using the principle of mathematical induction.

Total: 100 points