CMPUT 272 Final [B2 -- Harms]
April 24, 2001

NAME: ______________________________ Id: ______________________

Instructions:

- Write your name and student id number in the spaces above. Have your identification ready on the desk in front of you.
- Place all answers in the space provided on the question page. If you run out of room, write the answer on the back of a page.
- There are 6 questions and 8 pages. Check that your exam book is complete. The mark distribution is given beside the questions. The total number of marks is 60.
- This is an open book exam -- you are allowed to use the 2 textbooks, your notes and your assignments and exams that were given in this section this term. No other materials are allowed. No calculators or other such devices are allowed.
- You can use any of the notation we used in class (mathematical or Mizar notation or a combination) providing it is clear to me what you mean.
- This must be your individual work.
- Read each question carefully. Think before you answer! Show your work! Good Luck!

Question 1 [10 marks]

Consider the following 2 predicates: W[Alice, Bob] means that Alice works for Bob and P[Jim, Ann] means that Jim pays Ann.

a) Give the English meaning of the following logical statement
∃x (W[x,Bob]) implies ∀x (P[Bob,x]).

b) Give a logical statement that means "if no one pays Jim then everyone works for Jim".

c) Complete the following proof. Include one step for each rule of inference applied and mark in a comment what rule was used. Do not leave out any steps. You can use derived rules as well as basic rules.

environ

reserve x, y for Person;

A1: ∃ y ∀x W[x,y];

A2: ∀x,y (W[x,y] implies P[y,x]);

begin == prove that ∀x ∃ y P[y,x];

Question 2 [10 marks]

Consider the following program:

```
Nat n;
Nat A[1..n]
        Preconditions: n > 0
Nat k;
Boolean flag;

k, flag := 1, true;
do
        Variant?: n – k – ord(flag)
        Invariant?:
   [] flag & k < n  →  if
                          [] A[k] ≤ A[k + 1] →   k := k + 1;
                          [] A[k] > A[k+ 1] →   flag := false;
                     fi
   [] not flag  or  k = n  → exit
od
Postcondition: ?
```

Part a [2 points]: Explain why the variant given above is incorrect. Give a correct variant for this loop. Note that ord(flag) is 1 if flag is true and 0 if flag is false.

Part b [2 points]: What is the postcondition for this program?

Part c [6 points]: Find an invariant for the loop that verifies the postcondition and prove that it is always true.

Question 3 [10 marks]

Suppose we have the following binary relation: R: Nat ↔ Nat
where ∀ x,y (<x,y> ∈ R iff x ≠ y)

For each property defined below, either prove that the property holds for relation R by providing a formal proof or prove that the property does not hold for relation R by giving a counterexample.

a) Property 1 holds if ∀x,y,z ((<x,y> ∈ R & <y,z> ∈ R) implies (<x,z> ∈ R))
Note that this is the transitive property.

b) Property 2 holds if ∀x,y (<x,y> ∈ R implies <y,x> ∈ R)
Note that this is the symmetric property.

c) Property 3 holds if $\forall x,y \; (\langle x,y\rangle \in R$ implies $\exists z \; (\langle y,z\rangle \in R \;\&\; \langle x,z\rangle \in R))$

Question 4 [10 marks]

Consider the following 3 functions:

f: Int → Int where f(x) = x + 2
g: Int → Nat where g(x) = 3 x^2
h: Int → Int where f(x) = x - $\lceil 100 / (x + 1) \rceil$

Note that $\lceil y \rceil$ gives the "ceiling" of y. This is the smallest integer larger than or equal to y. For example, $\lceil 2.1 \rceil$ is 3.

Fill in the blanks in the table below. For each function, give the domain and put YES under the property column if that property holds for this function or NO if the property does not hold. Note that for composition f;h is the same as h(f(x)).

Function	Domain	Total	Onto	1-1
f				
g				
h				
f;f				
g;f				
f;h				
h;f				

Question 5 [10 marks]

Below are 2 universally quantified statements about sets. Determine whether each statement is true or false. Provide a formal proof if the statement is true using the definitions for set intersection and subset given below. Provide a counter-example if the statement is false (that is, give specific sets which make the statement false).

IntersectDef: $\forall X,Y\ (\forall x\ (x \in X \cap Y \text{ iff } x \in X \,\&\, x \in Y))$

SubsetDef: $\forall X,Y\ (X \subseteq Y \text{ iff } \forall x\ (x \in X \text{ implies } x \in Y))$

a) $\forall X,Y,Z\ (X \subseteq Z \rightarrow (X \cap Y \subseteq Y \cap Z))$

b) $\forall X,Y,Z\ ((X \cap Y \subseteq Y \cap Z) \rightarrow X \subseteq Z)$

Question 6 [10 marks]

Use induction to prove the following:

$$\forall n \left(\sum_{i=0}^{n} (4i + 2) = 2(n+1)^2 \right)$$