Professor: Thomas Harke

CMPUT 272 Fall 2000: Section A2
Final
Tuesday, Dec. 12, 14:00
Time: 120 minutes
Total Points 60

Last name:
First name:
Student number:

- This exam is **open book**
- This exam should have 8 pages and 9 questions. You are responsible for checking that your exam booklet is complete.
- Unless otherwise specified, you may use whichever of HR or GT notation you prefer.
- If you need additional space to write you may use the backs of the exam pages. If necessary, additional paper is available from a proctor.

Question	Mark	
1		8
2		2
3		7
4		8
5		8
6		4
7		7
8		6
9		10
Σ		60

Question 1 [8 points]:
In this question use only the basic rules of inference (namely NE, NI, CE, CI, DE, DI, IE, II, EqE, EqI, RE, ContrI). Each step requires a justification (i.e. a **reference** to the formulas it is inferred from) and an annotation (i.e. the **name** of the inference rule used).

Given the premises:

```
premise0:  not A[] implies not C[];
premise1:  B[] implies not D[];
```

Derive the conclusion:

```
(not A[] or B[]) implies not (C[] & D[])
```

Question 2 [2 points]:
Explain in words the difference between the statements $A \subseteq B$ and $A \in B$

Question 3 [7 points]:

Use induction to prove that for all natural numbers n the following holds:

$$\sum_{i=1}^{n}(6i - 2) = n(3n + 1)$$

Clearly state what your base case, induction step, and inductive hypothesis are, as well as where you use the inductive hypothesis.

Question 4 [8 points]:
Using `mizar`-like derivations, show that $A - (B \cup C) \subseteq (A - B) \cap (A - C)$. You may use derived rules of inference such as De Morgan's law, and you may combine steps that do not involve quantifiers.

```
environ
  reserve P, Q for SET, x for ELEMENT;
  SubsetDef: ∀P,Q [(P ⊆ Q) ⇔ (∀x (x ∈ P ⇒ x ∈ Q))];
  DiffDef:   ∀P,Q [∀x (x ∈ P − Q ⇔ (x ∈ P ∧ x ∉ Q))];
  UnionDef:  ∀P,Q [∀x (x ∈ P ∪ Q ⇔ (x ∈ P ∨ x ∈ Q))];
  InterDef:  ∀P,Q [∀x (x ∈ P ∩ Q ⇔ (x ∈ P ∧ x ∈ Q))];
  given A, B, C being SET;

begin
  claim:  A − (B ∪ C) ⊆ (A − B) ∩ (A − C)
  proof:
```

Question 5 [8 points]:
Translate the following sentences into predicate logic using the predicates and constants:

> **Good(x):** indicates that x is good.
> **Child(x):** indicates that x is a child.
> **GivesGift(x,y):** indicates that x will give a gift to y.

The universe of discourse consists of people, including Virginia and Santa.

1. Virginia is a good child but Santa will give not her a gift.

2. Santa will give a gift to every good child.

3. There is no bad child to whom Santa will give a gift.

4. Not everybody gets a gift from someone.

Question 6 [4 points]:
Show that the following proposition is a contingency by providing appropriate interpretations.

$$(\exists x\ P(x)) \Rightarrow (\forall x\ P(x))$$

Question 7 [7 points]:
The following procedure takes an array A as input, it modifies the contents of A, and then returns the new A as well as the index *sep*.

Mystery Program

```
Nat n, sep;
Rat A[1..n];
  Preconditions: n > 0
Nat l, r;
l, r := 1, n;
do
     Variant: ?
     Invariant: ?
   [] l = r + 1 --exit-->
   [] l < r + 1 -->
       if
          [] even(A[r]) --> r := r - 1;
          [] odd(A[l]) --> l := l + 1;
          [] odd(A[r]) & even(A[l]) -->
               A[l], A[r] := A[r], A[l];
               r, l := r - 1, l + 1;
       fi
od
sep := l;
  Postconditions: ?
```

1. State a variant for the loop.

2. In words, state the postcondition for this procedure.

3. Using formal notation, state the postcondition for this procedure.

4. In words, state the invariant for the loop

5. Using formal notation, state the invariant for the loop

Question 8 [6 points]:
This question concerns all functions f, g. Determine whether the following statements are true or false. Provide a counterexample for false claims. No justification is required for true claims. (Wrong answers count the same a missing answers :-))

(Recall that the order of composition is different for functions than relations: $(g \circ f)(x) = g(f(x))$.)

1. f is a bijection $\Rightarrow f^{-1}$ is a bijection.

2. $g \circ f$ is 1-1 $\Rightarrow g$ is 1-1.

3. $g \circ f$ is onto $\Rightarrow g$ is onto.

4. g is total $\Rightarrow g \circ f$ is total.

Question 9 [10 points]:
The following procedure takes values X and Y as input, and calculates X^Y.

Fast Exponentiation

Nat X, Y, res;
Preconditions: $n > 0$
Nat x, y;
$x, y, res := X, Y, 1;$
do
Variant: y
Invariant: $x^y \cdot res = X^Y$
$[]\ y = 0 \xrightarrow{exit}$
$[]\ odd(y) \longrightarrow y, res := y - 1, res \cdot x;$
$[]\ even(y)\ \&\ y > 0 \longrightarrow y, x := \lfloor \frac{y}{2} \rfloor, x \cdot x;$
od
Postconditions: $res = X^Y$

You may assume that we have already proven that the variant is nonnegative and decreasing, and that one of the guards always evaluates to true.

a) [8]: Prove that the invariant is true at the beginning of each iteration of the loop.

b) [2]: Prove that the postcondition is true when the loop exits.