Professor: Thomas Harke

CMPUT 272 Fall 2000: Section A1.
Final
Friday, Dec. 15, 9:00
Time: 120 minutes
Total Points 60

Last name:
First name:
Student number:

- This exam is **open book**
- This exam should have 8 pages and 9 questions. You are responsible for checking that your exam booklet is complete.
- Unless otherwise specified, you may use whichever of HR or GT notation you prefer.
- If you need additional space to write you may use the backs of the exam pages. If necessary, additional paper is available from a proctor.

Question	Mark	
1		7
2		4
3		7
4		9
5		8
6		3
7		7
8		5
9		10
Σ		60

Question 1 [7 points]:
This question require you to do a `mizar` derivation.

- Use only the basic rules of inference, namely NE, NI, CE, CI, DE, DI, IE, II, EqE, EqI, RE, ContrI.
- Each step requires a justification (i.e. a **reference** to the formulas it is inferred from) and an annotation (i.e. the **name** of the inference rule used).
- To save time writing, you may omit writing the "`[]`" after propositions

Given the premise:

```
premise:  (A[] or B[]) implies (not C[] & D[]);
```

Derive the conclusion:

```
(C[] or not D[]) implies not A[]
```

Question 2 [4 points]:
The truth table for the logical connective nor is:

π	ψ	π nor ψ
T	T	F
T	F	F
F	T	F
F	F	T

a) [1]: Why is the following elimination rule for nor sound: $\dfrac{\pi \texttt{ nor } \psi}{\texttt{not } \psi}$

b) [1]: Propose a sound introduction rule for nor.

Question 3 [7 points]:
Use induction to prove that $2^{4n} - 1$ is a multiple of 5 for all natural numbers n

Clearly state what your base case, induction step, and inductive hypothesis are, as well as where you use the inductive hypothesis.

Question 4 [9 points]:
Using `mizar`-like derivations, show that $(C \subseteq A \cup B) \Rightarrow (C - A \subseteq B)$. You may use derived rules of inference such as De Morgan's law, and you may combine steps that do not involve quantifiers.

environ:
  reserve P, Q for SET, x for ELEMENT;
  SubsetDef: $\forall P, Q\ [(P \subseteq Q) \Leftrightarrow (\forall x\ (x \in P \Rightarrow x \in Q))]$;
  DiffDef: $\forall P, Q\ [\forall x\ (x \in P - Q \Leftrightarrow (x \in P \wedge x \notin Q))]$;
  UnionDef: $\forall P, Q\ [\forall x\ (x \in P \cup Q \Leftrightarrow (x \in P \vee x \in Q))]$;
  given A, B, C being SET;

begin
  claim: $(C \subseteq A \cup B) \Rightarrow (C - A \subseteq B)$
  proof:

Question 5 [8 points]:
Translate the following sentences into predicate logic using the predicates:

Good(x): indicates that x is good.
Child(x): indicates that x is a child.
Parent(x,y): indicates that x is the parent of y.
GivesGift(x,y): indicates that x will give a gift to y.

The universe of discourse consists of people, including the constants **Virginia** and **Santa**.

1. Virginia is a bad child but Santa will give her a gift.

2. Virginia will give a gift to each of her parents.

3. Only good children will recieve gifts from Santa.

4. There is no good child to whom Santa will not give a gift.

Question 6 [3 points]:
Show that the following two propositions are not logically equivalent.

1. $[\exists x \; Q(x)] \Rightarrow [\forall x \; P(x)]$
2. $\exists x \; [P(x) \Rightarrow \forall y \; Q(y)]$

Question 7 [7 points]: The following procedure takes the array A and the value n as inputs, and returns the value res as output.

Mystery Program

Nat n;
Rat A[1..n], res;
Preconditions: (none)
Nat k;
Boolean valid;
$k, res, valid := 1, 0, true;$
do
    Variant: ?
    Invariant: ?
  [] $k = n + 1 \xrightarrow{exit}$
  [] **not** $valid \xrightarrow{exit}$
  [] $k \leq n$ & $valid \longrightarrow$
      if
        [] $A[k] = 0 \longrightarrow$
            $valid := false;$
        [] $A[k] \neq 0 \longrightarrow$
            $res := res + (1/A[k]);$
            $k := k + 1;$
      fi
od
Postconditions: ?

---

1. State a variant for the loop.

2. In words, state the postcondition for this procedure, relating the output to the inputs.

3. Using formal notation, state the postcondition for this procedure.

4. In words, state the invariant for the loop

5. Using formal notation, state the invariant for the loop

Question 8 [5 points]:
For each of the following functions, determine which of the following properties it exhibits. In each case, the domain and range spaces of the functions are $\mathbb{R}$. In the table below, write a "$\surd$" if the function satisfies the property, an "$\times$" if not, or leave it blank if you don't know.

Do not answer these unless you are fairly certain: marks will be deducted for wrong answers.

	total	injective	surjective
$\lambda x.e^x$			
$\lambda x.\ln(x)$			
$\lambda x.x^2$			
$\lambda x.\sqrt{x}$			
$\lambda x.x^3$			

Question 9 [10 points]:
The following procedure takes an array A as input, rearranges its elements so that all the odd numbers have lower indices than all the even numbers.

Partition by Parity

```
Nat n, sep;
Rat A[1..n];
Preconditions: n > 0
Nat l, r;
l, r := 1, n;
do
      Variant: 2 * (r + 1 - l) + ord(even(A[l]))
      Invariant: (∀i : 1 ≤ i < l : odd(A[i])) & (∀i : r < i ≤ n : even(A[i]))
   [] l = r + 1 --exit-->
   [] l < r + 1 -->
         if
            [] even(A[r]) --> r := r - 1;
            [] odd(A[l]) --> l := l + 1;
            [] odd(A[r]) & even(A[l]) --> A[l], A[r] := A[r], A[l];
         fi
od
sep := l;
Postconditions: (∀i : 1 ≤ i < sep : odd(A[i])) & (∀i : sep ≤ i ≤ n : even(A[i]))
```

You may assume that we have already proven that the variant is nonnegative and decreasing, and that one of the guards always evaluates to true.

a) [8]: Prove that the invariant is true at the beginning of each iteration of the loop.

b) [2]: Prove that the postcondition is true when the loop exits.